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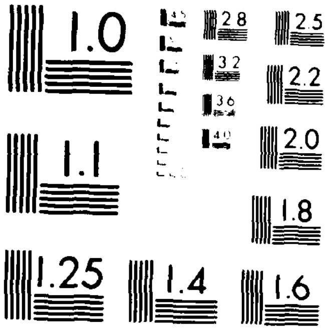

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THE CONSTRUCTION OF A PEDAGOGY TO
PROMOTE MEANINGFUL LEARNING OF
STRATEGIES AND TACTICS FOR SOLVING
PROBLEMS OF ELEMENTARY COMBINATORICS

THESIS

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THE CONSTRUCTION OF A PEDAGOGY TO PROMOTE MEANINGFUL LEARNING OF STRATEGIES AND TACTICS FOR SOLVING PROBLEMS OF ELEMENTARY COMBINATORICS

THESIS

Presented to the Faculty of the School of Systems and Logistics
of the Air Force Institute of Technology
Air University
In Partial Fulfillment of the
Requirements for the Degree of
Master of Science in Systems Management

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Preface

The purpose of this study was to develop a new method for teaching combinatorics, given the fact that previous presentations leave a great deal to be desired pedagogically. Most combinatorial textbooks use very difficult languages and notations, do not place enough emphasis on visualization and fail to show the relationships between general and particular concepts. Students, under these circumstances, usually learn combinatorics in a pure rote manner, finding no motivational reasons for this discipline.

A conceptual map was proposed in order to facilitate the teaching of this subject in a more graphical form, showing the hierarchical interrelations between superordinate and the subordinate concepts. Several examples of how such concepts can be visually taught have been created and presented, so that future research can investigate ways that computer graphics and expert systems can be used to facilitate the accession and employment of the conceptual map.

Along the process of this investigation I received constant support from my thesis advisor, Professor Daniel Reynolds, to whom I express in public my deep appreciation,

admiration and recognition. I also thank his wife, Phyllis Reynolds, for her extremely well done typing, that is not typing, but a piece of art. Finally, a big hug for my children Raquel, Pedro, Natalia and Patricia for their patient suffering of the bad moods of this part-time father and part-time mother.

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Abstract

The purpose of this study was to construct a new methodology for teaching combinatorics based on Doctor Ausubel's theory about meaningful learning. The key idea in Ausubel's theory is that if learning has to be meaningful, then the learner has to have subsuming or anchoring concepts in his cognitive structure.

Combinatorics has typically been one of those subjects the students have more difficulty in understanding. This phenomenon happens because previous presentations of combinatorics leave a great deal to be desired pedagogically, and do not place enough emphasis on visualization. As a result, students use to learn course materials in a rote manner, and find little motivation for such learning activities.

A prescription has been found to remedy such pathology. A conceptual map, rather than a typically organized hierarchy of concepts, has been developed. The conceptual map interrelates the main and subordinate concepts in a cyclical manner, in a repetitive way, in a gradual and smooth progress, to enable the reader to assimilate ideas meaningfully.

THE CONSTRUCTION OF A PEDAGOGY TO PROMOTE MEANINGFUL LEARNING OF STRATEGIES AND TACTICS FOR SOLVING PROBLEMS OF ELEMENTARY COMBINATORICS

I. Introduction

General Issue

Combinatorial analysis is an old branch of mathematical science which has been treated by many authors. Nevertheless, previous presentations leave a great deal to be desired pedagogically, and do not place enough emphasis on visualization. On the other hand, there is a lack of available course material and/or strategies to help the neophyte acquire an adequate combinatorial concept base to study more advanced notions of probability and statistics. These circumstances and needs are the motivation for the research study.

Problem Statement

The introduction of combinatorics in elementary statistics textbooks typically focuses on the memorization and subsequent regurgitation of nine key formulae that can be used to compute the total number of configurations of elements that meet certain criteria:

1. The number of variations, where order of the elements is relevant, and

2. The number of combinations, where order is irrelevant.

Because evaluation instruments and grading criteria associated with such learning activities normally reward students for rote learning, they promote a pedagogy that reduces, and sometimes eliminates, any opportunity for meaningful learning of course materials. As a result, concepts associated with counting problems appear dry and irrelevant. Management students, in particular, find little motivation for such learning activities and, at least during the period of studying such problems, even less utility.

The motivational vacuum fostered by such a pedagogy leaves facilitating the construction of an adequate combinatoric framework to serendipity, and encourages most students to begin their study of applied statistics without the necessary cognitive structures with which to assimilate, reconcile and integrate the concepts of probability theory. Without such structures, students, despite long hours of study, find it difficult to cultivate the skills needed to deal with the complexities of scientific research.

A pedagogy needs to be constructed that promotes a thorough diagnosis and an adequate supplementation of any shortfall in concepts associated with the central problems of combinatorics and management.

Research Objectives

The objectives of the research are as follows:

1. A conceptual mapping will be proposed rather than a topically organized hierarchy of concepts, that will facilitate the teaching of this subject, and allow concrete presentations of applications within the context provided by the conceptual map.

2. Several examples of how such concepts can be visually taught will be created and presented, so that future research can focus on their implementation on computer software. Combinatorial problems in one-, two-, and three-dimensional space will be introduced, but not exhausted, leaving an attractive open field for future investigations.

Such a renaissance in teaching strategy will encourage meaningful learning by establishing evaluation criteria and heuristics for learning how to learn that facilitate acquirement of the necessary cognitive structures for solving elementary combinatorial problems, and help students cultivate strategies and tactics for discovering solutions to more classical or possibly unique combinatorial problems.

A new pedagogy will be developed (employing typical problem scenarios accompanied by an explanation of the value of aforementioned heuristics) that practically

guarantees students (who choose to learn) can experience felt significance with course material.

Recent findings of educational psychologists favoring the theories of David Ausubel have demonstrated that, when such teaching and learning take place, the necessary conceptual foundations for whatever the subject, can be built by highly motivated and enthused students.

II. Research Background and Proposal

Introduction

Previous presentations of combinatorics leave a great deal to be desired pedagogically because most textbooks use difficult languages and notations, do not place enough emphasis on visualization, and fail to show the relationship between general and subordinate concepts. Students, under these circumstances, usually learn combinatorics in pure rote manner, finding no motivational reasons for this discipline.

Doctor Ausubel's theory about meaningful learning will be used in order to prescribe a remedy for such deficiencies. The key idea in Ausubel's theory is that "if learning is to be meaningful, then new knowledge to be learned must have anchoring concepts available in the learner's cognitive structure" (Novak, 1977:137).

This chapter is presented in four stages: three phases and a glossary.

Phase I will deal with Ausubel's theory of learning.

Phase II will present the reasons why current combinatorics texts simply do not support a pedagogy that facilitates the meaningful learning process defended by Ausubel.

Phase III will propose a new approach for teaching combinatorics by using a conceptual mapping, rather than a topically organized hierarchy of concepts, that will allow presentations of applications in one, two, and three dimensions.

Finally, a glossary provides definitions of the basic terms used in Ausubel's theory.

Phase I. Ausubel's Theory

Ausubel's theory of learning deals with four main concepts: meaningful learning versus rote learning and discovery learning versus reception learning.

Meaningful Learning. Novak defines meaningful learning as: "a process in which new information is related to an existing relevant knowledge (subsuming concepts) in the individual's cognitive structure" (Novak, 1977:74-75).

The cognitive structure of an individual refers to the total content and organization of this individual's ideas or concepts.

The subsuming or anchoring concepts are those relevant ideas that an individual has in his cognitive structure; those relevant ideas play the main role to enable the individual to assimilate new knowledge if there is any interrelationship between the new information and his previously existing ideas.

Rote Learning. Novak points out that

> When relevant concepts do not exist in the cognitive structure of an individual, new information must be learned by rote, a process in which new information is not associated with existing concepts in the individual cognitive structure, and therefore little or no interaction occurs between newly acquired information and information already stored [Novak, 1977:77].

It is important to distinguish between reception learning versus discovery learning too.

Reception Learning. Individuals receiving this type of learning play a passive role in the sense that they accept external elaborated information because it is presented in logical, assembled and congruent form; thus they can assimilate it into their cognitive structures. That is why Ausubel defines reception learning as "that kind of learning in which the entire content of what is to be learned is presented to the learner in more or less final form" (Ausubel and others, 1978:629). Ausubel emphasizes the role of reception learning in all education, especially in schools.

Discovery Learning. The individuals receiving discovery learning have to play an active role in the sense that the content to be learned has to be selected, discovered, acquired and digested by the learner. That is the reason why Ausubel defines discovery learning as

> . . . that kind of learning in which the content of what is to be learned is not given or presented, but must be discovered by the learner before he can assimilate it into his cognitive structure [Ausubel and others, 1978:626].

Reception learning and discovery learning are viewed as a separate and independent continuum from that involving rote learning and meaningful learning. Figure 1 shows typical forms of learning. An interesting explanation of Figure 1 is given by Novak when he says,

> Discovery learning can be rote; anyone can discover a solution to a puzzle or algebra problem by trial and error, and not associate this "discovery" with existing knowledge in cognitive structure. A "discovery" made by a scientist is not a real discovery until the new information can be related to concepts already familiar to scientists or to a new concept that encompasses or supercedes earlier concepts [Novak, 1977:100-101].

Ausubel's learning theory is extremely important for combinatorics educational purposes since most textbooks and presentations do not usually provide an easy path for meaningful learning. If the task of learning combinatorics is to be meaningful, then the new knowledge to be learned must have subsuming, anchoring ideas available in the students' cognitive structure. These subsuming ideas have to be the most general, main, basic and inclusive combinatorial concepts.

The lack of suitable course material promotes in students a rote approach to combinatoric learning, that is subsequently forgotten because "forgetting depends primarily on the degree of meaningfulness associated with the learning process" (Novak, 1977:84).

It is also very important to remark here the significance of the following paragraph written by Novak:

Meaningful Learning	Clarification of relationships between concepts	Well designed audio-tutorial instruction	Scientific research. New music or architecture
	Lectures or most textbook presentations	School laboratory work	Most routine "research" or intellectual production
Rote Learning	Multiplication tables	Applying *formulas* to solve problems	*Trial and error* "puzzle" solutions
	Reception Learning	Guided Discovery Learning	Autonomous Discovery Learning

Fig. 1. Typical Forms of Learning [Novak:1974:101]

> Most information learned by rote in schools is lost within six to eight weeks. As a result, students recognize that they have forgotten much of the information presented earlier, and that their earlier but now lost learning is interfering with new learning [Novak, 1977:85].

Phase II. Diagnosis

The pathology of current combinatoric textbooks is clear, according to Ausubel's theory, because it does not offer a pedagogical procedure that facilitates meaningful learning. Such approach to teaching should follow a congruent, assembled and logical model; for instance,

STAGE 1. RECEPTION LEARNING

1. Introduce the most general and inclusive concepts first.

2. Introduce the specific and subordinate concepts later.

STAGE 2. GUIDED DISCOVERY LEARNING

1. Show explicit relationships between the most general and inclusive concepts.

2. Present explicit relationships between the general and the subordinate concepts.

3. Show explicit relationships between all kinds of subordinate concepts.

Following this model, reconciliation of concepts should be best achieved, because instruction deals with all ideas at all levels of the conceptual hierarchy in an up and down cycle fashion.

A critical aspect that reinforces the pathology for teaching combinatorics is the nomenclature normally used in textbooks, that requires a deep background in some other disciplines, such as theory of groups, for instance, or many notions of what has been called modern mathematics. These supporting areas of knowledge are not completely required and should be avoided in a great percentage. Thus, combinatorics can be presented in a more suitable, easy and straightforward manner, without loss of continuity or content.

On the other hand, the lack of scientific rigor in those textbooks is astonishing when they extrapolate results from particular cases to general conclusions, following unclear inductive methods.

Examples of what has been criticized can be found all along the following main books about combinatorics published in the United States:

1. Elementary Combinatorial Analysis, by Martin Eisen. New York: Gordon and Breach, Science Publishers, 1969, from page 1 to page 215 (entire book).

2. Introduction to Combinatorial Mathematics, by Chang Laung Liu. New York: McGraw-Hill Book Company, Computer Science Series, 1968, from page 1 to page 161 (Chapters, 1, 2, 3, 4, and 5).

3. Elements of Discrete Mathematics, by Chang Laung Liu. New York: McGraw-Hill Book Company, Computer Science Series, 1977, from page 1 to page 45 (Chapters 1 and 2).

4. Notes on Introductory Combinatorics, by George Polya, Robert E. Tarjan, and Donald R. Woods. New York: Birkhauser, 1983, page 1 to page 115 (Chapters 1, 2, 3, 4, 5, 6, 7, and 8).

Under such a lack of pedagogy, opportunities for combinatoric meaningful learning is sometimes eliminated, and students are forced to study course material in a rote manner. In order to improve the above-mentioned situation, a prescription will be given in the following Phase III.

Phase III. Prescription

To promote meaningful learning along any stage of the continuum, from reception to discovery learning, a remedy will be provided by:

1. Developing the most general and inclusive concepts first, and the specific and subordinate concepts later, all of them assembled on logical sequential order of difficulty.

2. Illustrating some relationships between all concepts in order to make clear distinction between which one of those are the most general and superordinate, and which ones are more specific and subordinate.

3. Introducing a conceptual map, rather than the mere definition of the topics, that will graphically show those relationships bridging general and subordinate concepts. The conceptual map will be used recursively in solving combinatoric counting problems in one, two and three dimensions.

4. Presenting an adequate set of problems, arranged in increasing degree of difficulty, to help students understand the combinatorial hidden snags. Such problem-solving ability represents an opportunity for meaningful learning because it requires well-differentiated relevant concepts in cognitive structure.

Glossary (Ausubel and others, 1978:624-630)

1. Anchoring idea(s)--an established relevant idea (proposition or concept) in cognitive structure to which new ideas are related and in relation to which their meanings are assimilated in the course of meaningful learnings. As a result of this interaction they themselves are modified and differentiated.

2. Cognitive structure--the total content and organization of a given individual's ideas; or, in the context of subject-matter learning, the content and organization of his or her ideas in a particular area of knowledge.

3. Discovery learning--that kind of learning in which the principal content of what is to be learned is not given (or presented), but must be discovered by the learner before he can assimilate it into cognitive structure.

4. Forgetting--a process of memorial reduction or obliterative assimilation that occurs in the course of storage (retention); as a result of this process, the dissociability strength of an acquired meaning falls below the threshold of availability and the meaning is accordingly no longer retrievable.

5. Idea--a concept or proposition relatable to cognitive structure.

6. Integrative reconciliation--part of the process of meaningful learning that results in explicit delineation of similarities and differences between related ideas.

7. Meaning--a differentiated and sharply articulated content of awareness that develops as a product of meaningful symbolic learning or that may be evoked by a symbol or group of symbols after the latter have been nonarbitrarily and substantively related to cognitive structure.

8. Meaningful learning--the acquisition of new meanings; it presupposes a meaningful learning set and a potentially meaningful learning task (that is, a task that can be related in nonarbitrary, substantive fashion to what the learner already knows). Part of the rote → meaningful learning continuum as distinct from the reception → discovery learning continuum.

9. Meaningfulness--the relative degree of meaning associated with a given symbol or group of symbols as opposed to their substantive cognitive content, as measured by degree of familiarity, frequency of contextual encounter or degree of lexical substantiveness (for example, a noun or verb in contrast to a proposition).

10. Problem solving--a form of directed activity or thought in which both the cognitive representation of prior experience and the components of a current problem situation are reorganized, transformed, or recombined in order to achieve a designated objective; involves the generation of problem-solving strategies that transcend the mere application of principles to self-evident exemplars.

11. Progressive differentiation--part of the process of meaningful learning, retention, and organization that results in further hierarchical elaboration of concepts or propositions in cognitive structure from the top downwards.

12. Reception learning--that kind of learning in which the entire content of what is to be learned is presented to the learner in more or less final form. Related to the reception → discovery continuum as distinct from the rote → meaningful learning continuum.

13. Rote learning--the acquisition of arbitrary, verbatim associations in learning situations where either the learning material itself cannot be nonarbitrarily and substantively related to cognitive structure, or where the learner exhibits a nonmeaningful learning set.

14. Superordinate learning--learning the meaning of a new concept or proposition that can subsume relevant and less inclusive particular ideas already present in cognitive structure.

III. Presentation of Concepts

Introduction

This chapter will introduce the most general and inclusive concepts of combinatorics first, and the specific and subordinate concepts later, permitting a smooth reception process into the students' cognitive structure. Later on, those concepts will be explained in detail and graphically represented, allowing a guided discovery learning when some considerations will be made about the relationships between concepts compiled in a hierarchical conceptual map.

Thus, three interconnected sequences are going to be presented in an uninterrupted flow:

1. Sequence 1--main and subordinate concepts
2. Sequence 2--graphical representation of concepts
3. Sequence 3--relationships between concepts

Along the trip that goes from reception learning to discovery learning (guided), students should be able to find enough clearness, hopefully, for a meaningful learning task.

Sequence 1

Main Concepts. Combinatorics deals ith the problem of counting how many different subsets can be made by choosing n elements from a given set of m distinct objects.

In other words, given a set of m different elements ($a_1 \neq a_2 \neq a_3 \neq . . . \neq a_m$), combinatorics' job consists of computing the number of all possible distinct selections of n objects that, meeting certain criteria, can be made from the given original set of m different elements. Such criteria only have two options:

Option 1--the order is relevant (variations)

Option 2--the order is irrelevant (combinatorics)

Option 1--Order is Relevant (Variations). Two subsets of n elements selected from m, such that both subsets contain exactly the same elements, are considered different subsets if the order their elements are arranged is different.

Example. Given a set (a, b, c, d), two different subsets are (a, b, c) and (a, c, b) because even though both have the same elements (a, b, c), they are arranged in different order. When order is the relevant criteria, the combinatorics notion is named variations.

Consequently, variations can now be defined like this: any ordered sequence of n objects taken from a set of m distinct objects is called variation of size n.

Another way of expressing the same idea could be: Given a set of m distinct elements $a_1 \neq a_2 \neq a_3 \neq . . . \neq a_m$, variation of size n is every possible subset composed by n elements taken from those m, agreeing that two

variations of size _n_ are different if they have at least one element that is different; or if both have the same elements, these are arranged in a different order. That is to say, subsets are different if they differ at least in one element, or, if they have the same elements, their arrangement (order) is different.

Example. The twenty-four variations of the four different elements (a, b, c, d) taken in subsets of three elements are:

(a,b,c), (a,c,b), (b,a,c), (b,c,a), (c,a,b), (c,b,a),
(a,b,d), (a,d,b), (b,a,d), (b,d,a), (d,a,b), (d,b,a),
(a,c,d), (a,d,c), (c,a,d), (c,d,a), (d,a,c), (d,c,a),
(b,c,d), (b,d,c), (c,b,d), (c,d,b), (d,b,c), (d,c,b).

One way to symbolize this accounting problem is $V_4^3 = 24$. In general, V_m^n would represent variations of _m_ elements taking _n_ at a time.

In Chapter IV, a general procedure for calculating variations will be explained.

Option 2--Order is Not Relevant (Combinations). Given a set of _m_ distinct objects, any unordered subset of size _n_ is called _combination_. Thus, combinations of those _m_ different elements taken in subsets of length _n_ have to differ in one element at least.

Example. If three soldiers have to be chosen from a group of 4, the C_4^3 are four:

(a,b,c), (a,b,d), (a,c,d), (b,c,d), because here order of the elements does not matter.

In Chapter IV a general formula will be deducted for computing combinations of $\underline{m}$ different elements taken $\underline{n}$ at a time.

So far, the two basic concepts of variations and combinations have been introduced, albeit from these main ideas some specific and subordinate concepts should be derived.

Subordinate Concepts.

Option 1--Order is Relevant (Variations). Given the initial set of $\underline{m}$ distinct objects ($a_1 \neq a_2 \neq a_3 \neq . . . \neq a_m$) the selection of subsets of size $\underline{n}$ can be constructed in two forms:

1. No permitting replacement into the original set of length $\underline{m}$ of any element that has been used for constructing a subset of length $\underline{n}$. In this manner, objects cannot be repeated because they cannot go back and forth from the original set into any subset, and vice versa. It is a one-way trip. Thus, $m > n$ (if $m = n$, this is a particular case called permutations, that will be studied later). These kinds of variations are named ordinary variations, the notion having already been introduced. Remember that, for example, $V_4^3 = 24$.

2. But if replacement is permitted and, consequently, all elements belonging to the original set can be repeated any amount of times in the process to construct the subsets of length n, the problem is absolutely different, and this new situation could be seen as if the original set has now m different elements, each of them indefinitively repeated. This is the case called variations with repeated selection.

Example. The nine different two-digit numbers that can be formed using the single digits 1, 2, and 3 are:

12, 13, 23, 11, 22, 33, 21, 31, 32.

A form to represent the variations with repeated selection for this problem is $VR_3^2 = 9$.

In general, VR_m^n represents variations with repeated selection of m elements taken n at a time.

3. Going back to the concept of ordinary variations (where replacement is not allowed) it has been assumed that $m > n$; that is to say, the number of the different objects composing the original set is greater than the number of the different elements belonging to every subset. But nothing prevents that, in the limit, m could be equal to n. In such case, every subset of length m has to take all the m elements belonging to the original set. As in variations the order of the elements is relevant, every subset of length m has to have a different arrangement. In this particular case, ordinary variations

are called ordinary permutations, and they are represented as $V_m^m = P_m$.

Example. The six permutations that can be formed with the elements (a, b, c) are:

(a,b,c)

(a,c,b)

(b,a,c)

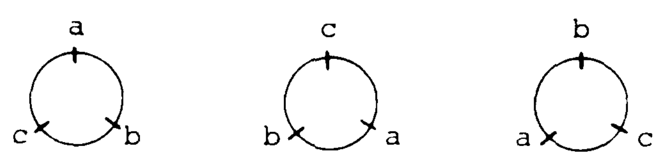
(c,b,a) $\qquad P_3 = 6$

(c,a,b)

(b,c,a)

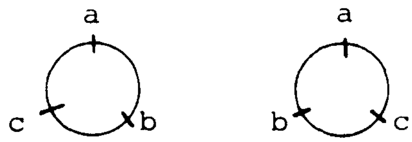
4. Speaking about ordinary permutations as a particular case of ordinary variations when the number n of objects composing each subset coincides with the length m of the original set (m = n), some extremely important considerations have to be made:

a. In two dimensions. For instance, when an observer is trying to solve on a piece of paper (that is a plane) the problem of counting in how many different ways he can arrange (that means, permute) three different elements (a, b, c) in a circular fashion, he would notice that



These three permutations are all the same because the relative position of the three elements is always the same;

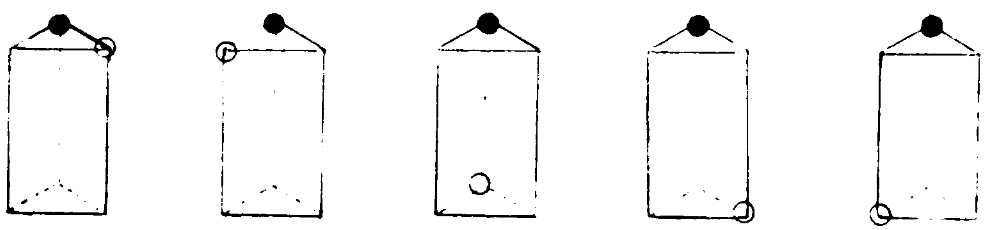
so, instead of three, he finds there are only two different permutations:



Thus, if every single linear ordinary permutation is going to be seen three times circularly, the observer should divide between 3 the number of ordinary permutations in order to solve his problem. If $P_3 = 6$, now $\frac{6}{3} = 2$ is the solution.

This case is called circular permutations (PC), and the ability to visualize it well will be extremely useful later on in order to solve three-dimensional problems.

b. In three dimensions, the observer is trying to figure out, for example, in how many different ways he can paint the vertices of a right equilateral triangular prism, painting one vertex in red, one vertex in green, and leaving blank the other four. The graphical solution is



This is a more difficult problem, but especially is going to deal with circular permutations by rotating the figures along all these axes of symmetry and computing only the different arrangements (configurations or permutations)

that appear after every rotation (eliminating those arrangements that are identical). In this process, the concept of substitutions will be presented later. This is a method that facilitates the representation of the relative position of the vertices in the space every time the figure (prism, tetrahedron, pyramid, cube, etc.) has been rotated a certain amount of degrees.

Going back to the problem, let us say for now that the solution is five, as can be seen graphically.

5. In the way combinatorics concepts have been presented, an assumption has been made: all m elements belonging to the original set are different; thus, $a_1 \neq 2_2 \neq a_3 \neq \ . \ . \ . \neq a_m$. But a new situation can be presented if several of those m elements are repeated. Under the new assumption, the original set structure could be

$$\overbrace{\underbrace{a_1 a_1 \ldots a_1}_{\alpha} \underbrace{a_2 a_2 \ldots a_2}_{\beta} \underbrace{a_3 a_3 \ldots a_3}_{\gamma} \ldots \underbrace{a_{\cdot} a_{\cdot} \ldots a_{\cdot}}_{}}^{m}$$

where element a_1 is repeated α times,

element a_2 is repeated β times,

element a_3 is repeated γ times,

. . . .

. . . .

element a[illegible] is repeated [illegible] times,

being $\alpha + \beta + \gamma + \ldots +$ [illegible] $= m$.

When calculations are made for computing the number of different permutations here, some corrections to the concept of permutations formerly presented have to be made to avoid the selection of duplicated arrangements.

Example. The different ways that the three letters a, b, b can be permuted are three:

(a,b,b), (b,a,b) and (b,b,a),

a solution that is different to be one presented in the previous paragraphs.

Thus, a new notion has to be introducted, that constitutes an exemption to the general hypothesis that $a_1 \neq a_2 \neq a_3 \neq \ldots \neq a_m$. The name for the new concept is permutations with repetition, and will be a golden key that will unlock the door of many problems.

Option 2--Order is Not Relevant (Combinations). Given the initial set of m distinct objects $(a_1 \neq a_2 \neq a_3 \neq \ldots \neq a_m)$, the selection of subsets of size n can be constructed in two forms:

1. No permitting replacement into the original set of length m of any element that has been used for constructing a subset of length n. Thus, objects cannot go back and forth from the original set to the subsets. Thus, $m > n$. These kinds of combinations are called ordinary combinations, the notion of which has already been explained in option 2. Recall that $C_4^3 = 4$.

2. But if replacement is permitted and, consequently, elements comprising the original set can be repeated in constructing the subsets, the case is totally different, and the new situation can be seen as if the original set has now $\underline{m}$ different numbers, each of them indefinitely repeated. This case is named combinations with repeated selection.

Example. With the three objects a, b, c, the following combinations with repeated selection can be made:

Length 2. aa, ab, ac, bb, bc, cc.

Length 3. aaa, aab, aac, abb, abc, acc, bbb, bbc, bcc, ccc

Length 4. aaaa, aaab, aaac, aabb, aabc, aacc, abbb, abbc, accc, bbbb, bbbc, bbcc, bccc, cccc

A form to represent this problem is

$$CR_3^2 = 6; \; CR_3^3 = 10; \; CR_3^4 = 15$$

In general, CR_m^n.

<u>Sequence 2</u>

Main and subordinate concepts can be interrelated in the hierarchical map shown in Figure 2.

<u>Sequence 3</u>

1. The highest and most inclusive concept in combinatorics is ordinary variations because it takes into account every possible different configuration of the elements attending to the order in which the chosen elements are arranged.

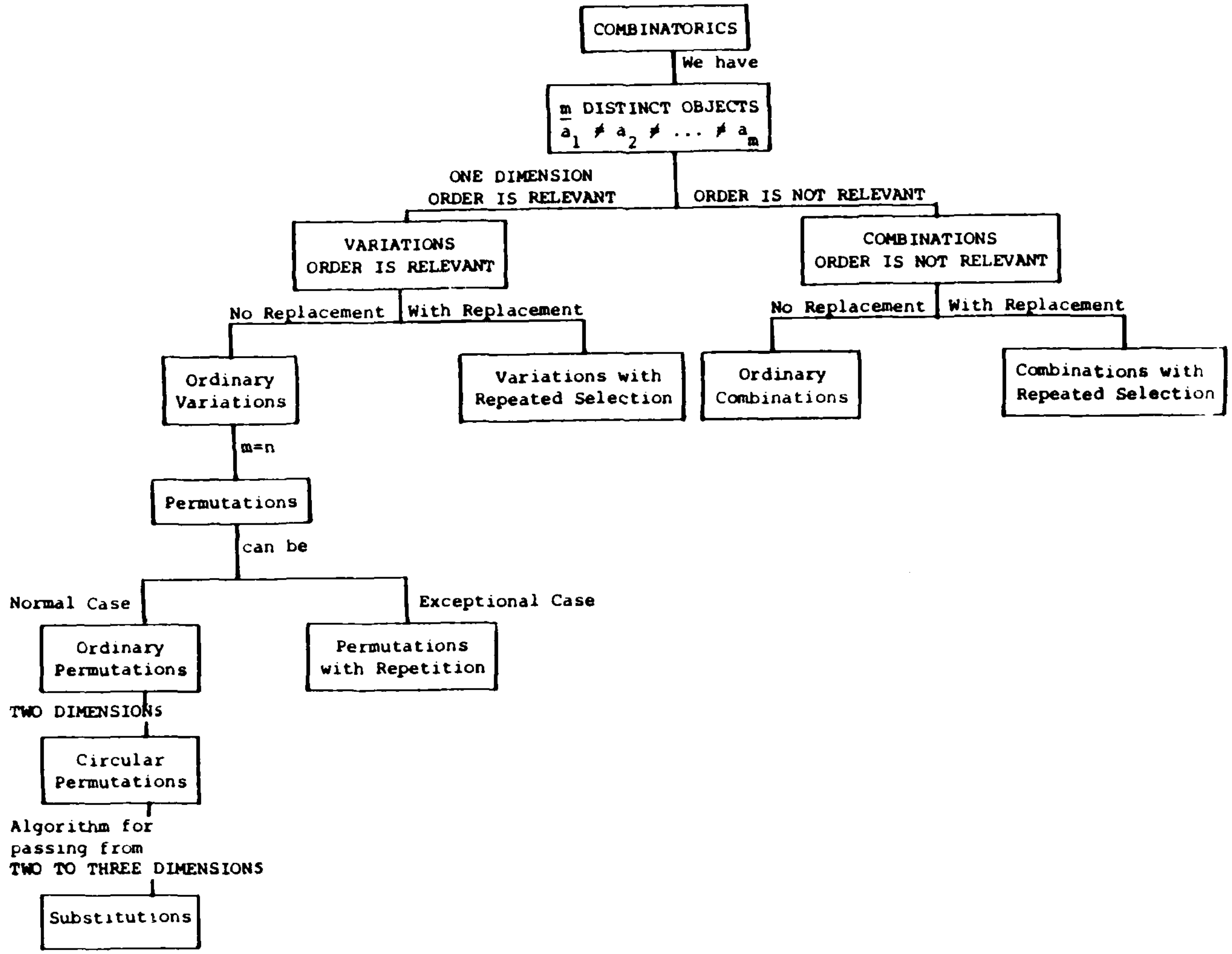


Fig. 2. Conceptual Map

In a second level of importance are ordinary combinations, where order of the chosen elements is irrelevant. For instance, if 234 is a particular combination of length 3, all the possible arrangements (permutations) that can be made with these three digits are

234, 243, 423, 324, 342 and 432.

These six different numbers represent the same combination, but they are six different variations. Thus, if the n elements of a combination are permuted in all possible ways, the result is variations. That means

$$C_m^n \cdot P_n = V_m^n$$

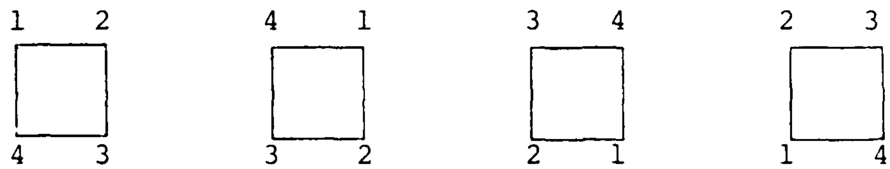
2. Ordinary permutations is a particular case of ordinary variations, when m = n. Thus, $V_m^n = P_m$.

3. Permutations with repetition should be considered particular cases of ordinary variations when the general hypothesis that $a_1 \neq a_2 \neq a_3 \neq \ldots \neq a_m$ is broken and some elements are equal between them.

4. Circular permutations concept is the same as ordinary permutations concept. What introduces a difference here is the fact that ordinary permutations are developed in one spacial dimension, while circular permutations are presented in two spacial dimensions. Thus, the observer counts differently in every case. For instance,

$$
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3
\end{array}
$$

represent four different ordinary permutations, but only one circular permutation



because the spectator has the freedom in two dimensions to rotate or flip the figure along its axes of symmetry and verify that the relative position of the four elements is always the same.

All these relationships will be explicitly shown in Chapter IV, where formulas to be applicable to every concept already introduced will be deduced, followed by some practical applications.

IV. Formulas and Applications

Introduction

This chapter will be divided into the three following sections:

1. Section I. Deduction of the general formulas of:

   a. Ordinary variations.

   b. Variations with repeated selection.

   c. Ordinary permutations.

   d. Circular permutations.

   e. Permutations with repetition.

      (1) Power of the polynomial (Leibniz' Formula).

   f. Ordinary combinations.

      (1) Properties.

      (2) The Tartaglia's Triangle.

      (3) Power of a binomial (Newton's Formula).

2. Section II. Solution of concrete problem scenarios in one and two dimensions using the conceptual framework provided by the concept map.

3. Section III. This part will exclusively deal with solving combinatoric problems in three dimensions. When a regular body is rotated or flipped in the space along its axes of symmetry, students have to be able to

count how many different arrangements (permutations) of the faces, vertices and sides, they can see.

As it will be seen later, permutations in one dimension (Ordinary Permutations) compute a bigger number of possibilities than permutations in two dimensions (Circular Permutations), and these compute an even bigger number of possibilities than permutations in three dimensions (Substitutions). The conclusion is that as more freedom a body has, the simplest is the solution (less possibilities), which goes against human intuition.

The theory of substitutions (Ordinary and Circular Substitutions) will be presented as a method that will allow any static observer to track the relative interposition of the vertices, faces and sides of a regular body when it is rotated symmetrically in the space.

In order to solve this kind of problem, two steps are necessary:

Step 1. Calculate the cycle index. The cycle index (that serves as a catalyst) consists in a division or proportion between:

a. Numerator: the total number of rotations performed, grouping them in circular substitutions or cycles that have the same pattern or structure.

b. Denominator: the total number of rotations performed.

For instance, if the total number of rotations that has been performed is 10, and these 10 rotations can be classified in three different kinds of circular substitutions, such that

10 = 1 circular substitution class A + 3 circular substitutions class B + 6 circular substitutions class C, the cycle index will be:

$$\frac{1\ \begin{array}{c}\text{circular}\\ \text{substitution}\\ \text{class A}\end{array} + 3\ \begin{array}{c}\text{circular}\\ \text{substitutions}\\ \text{class B}\end{array} + 6\ \begin{array}{c}\text{circular}\\ \text{substitutions}\\ \text{class C}\end{array}}{10}$$

Step 2. Calculate the pattern inventory. This tool consists of the replacement of every term in the nominator of the cycle index (circular substitutions class A, circular substitutions class B, etc.) by all the possible circular substitutions belonging to each class. Really, what the pattern inventory does is the development in detail of the cycle inventory. Hopefully, analyzing the pattern inventory, an answer to every specific problem should be deducted, as it will be seen later.

Section I. General Formulas

Ordinary Variations (Order is Relevant). Any ordered sequence of n objects taken from a set of m distinct objects is called a variation of size n of the objects.

Given $\underline{m}$ different elements $a_1 \neq a_2 \neq a_3 \neq \ldots \neq a_m$, variation of size $\underline{n}$ is any group of $\underline{n}$ elements chosen from those $\underline{m}$, agreeing that two variations are different <u>if they differ in one element at least</u>, or if both have the same elements, <u>their order of allocation is different</u>. Thus, two variables are taken into account in variations: different elements and order of the elements.

The number of variations of size $\underline{n}$ made from $\underline{m}$ given elements, will be symbolized by V_m^n. Notice that here $m > n$.

The general formula for ordinary variations can be deduced in this way:

Supposing that all possible variations of size $(n-1)$ are known (V_m^{n-1}), there are $m - (n-1) = (m-n+1)$ elements that have not been used in every variation of size $(n-1)$.

The objective now is passing from V_m^{n-1} to V_m^n. If every one of the $(m-n+1)$ elements that have not been used in every V_m^{n-1} is added to its right, for each variation, V_m^{n-1} will appear $(m-n+1)$ variations of the type V_m^n.

Giving values 1, 2, 3, ..., n to $\underline{n}$ in the expression $(m-n+1) \cdot V_m^{n-1} = V_m^n$,

$$n = 1, \quad (m-1+1)V_m^{1-1} = mV_m^0 \quad = V_m^1$$

$$n = 2, \quad (m-2+1)V_m^{2-1} = (m-1)V_m^1 \quad = V_m^2$$

$$n = 3, \quad (m-3+1)V_m^{3-1} = (m-2)V_m^2 \quad = V_m^3$$

$$n = 4, \quad (m-4+1)V_m^{4-1} = (m-3)V_m^3 \quad = V_m^4$$

$$n = n, \quad (m-n+1)V_m^{n-1} = (m-n+1)V_m^{n-1} = V_m^n$$

Therefore,

$$V_m^1 = m$$

$$V_m^2 = (m-1)V_m^1$$

$$V_m^3 = (m-2)V_m^2$$

$$V_m^4 = (m-3)V_m^3$$

$$V_m^n = (m-n+1)V_m^{n-1}$$

Multiplying these qualities member by member and eliminating common factors,

$$V_m^n = m(m-1)(m-2)(m-3) \quad \ldots \quad (m-n+1) = \frac{m!}{(m-n)!}$$

$$V_m^n = \frac{m!}{(m-n)!}$$

Example. How many different numbers of three characters could be formed with the digits 1, 2, 3, and 4 without repeated selection?

$$V_4^3 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 4.3.2.1 = 24 \text{ numbers.}$$

Example. How many different numbers of four digits without having a repeated digit are there in the decimal system?

The total different numbers of size four that can be generated with the ten digits (0, 1, 2, ..., 9) are V_{10}^4, but those beginning with 0 have to be subtracted because they are not four-digit numbers. Therefore,

$$V_{10}^4 - V_9^3 = \frac{10!}{(10-4)!} - \frac{9!}{(9-3)!} = \frac{10!}{6!} - \frac{9!}{6!} = \frac{1}{6!}(10!-9!)$$

$$= 4{,}536 \text{ numbers.}$$

Variations with Repeated Selection (Order is Relevant). If in ordinary variations every variation has to be composed by different elements, in variations with repeated selection it is possible that any object a_1, a_2, a_3, ..., a_n could be repeated any number of times.

The number of variations with repeated selection of size n, made from a set of length m, will be symbolized by VR_m^n. Notice here that m><n.

The general formula for variations with repeated selection can be deduced in this way:

Suppose that all possible variations with repeated selection of size (n-1) are known (VR_m^{n-1}). There are always m elements available to use when the objective is passing from VR_m^{n-1} to VR_m^n.

If every one of the $\underline{m}$ elements that have not been used to construct every VR_m^{n-1} is added to its right, for each variation with repeated selection VR_m^{n-1} will appear $\underline{m}$ variations with repeated selection of the type VR_m^n. Thus, if one variation VR_m^{n-1} produces $\underline{m}$ variations VR_m^n, all variations VR_m^{n-1} will produce $m \cdot VR_m^{n-1} = VR_m^n$. Giving values 1, 2, 3, ..., n in the expression $m \cdot VR_m^{n-1} = VR_m^n$,

$$n = 1, \quad mVR_m^{1-1} = mVR_m^0 = VR_m^1 = m$$

$$n = 2, \quad mVR_m^{2-1} = mVR_m^1 = VR_m^2$$

$$n = 3, \quad mVR_m^{3-1} = mVR_m^2 = VR_m^3$$

$$n = 4, \quad mVR_m^{4-1} = mVR_m^3 = VR_m^4$$

$$n = n, \quad mV_m^{n-1} = mVR_m^{n-1} = VR_m^n$$

Multiplying these equalities member by member and eliminating common factors $VR_m^n = m^n$.

Example. How many different numbers of three characters could be formed with the four digits 1, 2, 3, and 4?

$$VR_4^3 = 4^3 = 64 \text{ numbers.}$$

Example. How many different numbers of four digits are there in the decimal system?

$$VR_{10}^4 - VR_{10}^3 = 10^4 - 10^3 = 9000.$$

Ordinary Permutations (Order is the only Possible Variable. In ordinary permutations, order of the elements is not only important, it is the only variable that can be taken into account because the same m elements are always selected from the original sample space. Thus, different permutations can only differ in the way those m elements are arranged. Recall that two variables were taken into account when n elements were selected from m for constructing variations.

Ordinary permutations are a particular case of ordinary variations when m = n. Thus, the ordinary variations of size m that are made by choosing all the elements belonging to the original set of m objects are called ordinary permutations. That is to say

$$P_m = V_m^m = \frac{m!}{(m-m)!} = \frac{m!}{0!} = \frac{m!}{1} = m!$$

Mathematicians have convened that 0! = 1 because if $V_m^n = m(m-1)(m-2) \;.\;.\;.\; (m-n+3)(m-n+2)(m-n+1)$,

Therefore,

$$V_m^m = m(m-1)(m-2) \;.\;.\;.\; (m-m+3)(m-m+2)(m-m+1)$$

$$= m(m-1)(m-2) \;.\;.\;.\; 3 \cdot 2 \cdot 1 = m!$$

Thus,

$$P_m = m! \qquad \text{Q.E.D.}$$

Example. In how many different ways can the seven colors of the rainbow be rearranged?

$P_7 = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5,040$ ways.

Example. How many different words of eight letters can be constructed with the vowels a, i, o, u, and the consonants b, c, d, f, without having two vowels or two consonants together, and without repeating any vowel or consonant?

The four vowels can be arranged in $P_4 = 4!$ ways.

The four consonants can also be arranged in $P_4 = 4!$ ways.

Every arrangement of the 4! permutations made with vowels can be associated with every arrangement of the 4! permutations of consonants in two different ways: beginning with vowel or beginning with consonant.

For example, $\underline{a}$ b $\underline{u}$ c $\underline{i}$ f $\underline{o}$ d or

b $\underline{a}$ c $\underline{u}$ f $\underline{i}$ d $\underline{o}$

Therefore, $P_4 \cdot P_4 \cdot 2 = 2 \cdot (P_4)^2 = 2(4!)^2 = 1,152$ different words.

Circular Permutations (Ordinary Permutations but in a Two-dimensional Space). The difference between ordinary permutations and circular permutations is that the former works in one dimension (a line), but the later works in

two dimensions (a plane). For instance, given three people a, b, c, two different problems are:

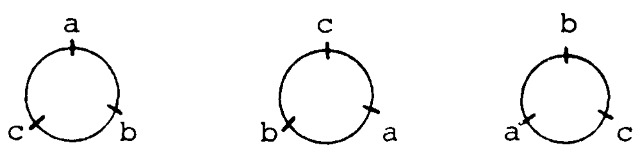
1. Arrange the three people in a row in all possible ways (ordinary permutations); and

2. Arrange the three people around a table in all possible ways (circular permutations).

The solution for the first problem is easy, $P_3 = 3! = 6$ different arrangements. Graphically,

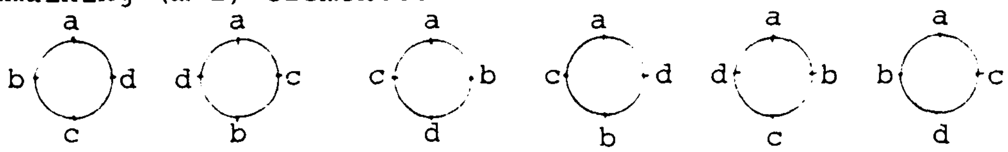
abc, acb, bac, bca, cab, cba

But in two dimensions the problem is not so easy because here there is no first and no last element. What counts is the relative position of every object in respect to the other objects. Thus, three arrangements



are indeed the same circular permutation because the intermutual position of the three elements is identical. Therefore, the problem has only two solutions in two spacial dimensions:

A general procedure for forming the number of circular permutations that can be done with m different objects ($a_1 \neq a_2 \neq a_3 \neq \ldots \neq a_m$) could consist of fixing one element and permuting, by ordinary permutations, the remaining (m-1) elements.



Given four different elements (a,b,c,d), the figure above represents the six different circular permutations that can be formed. Element a has been fixed, and the other three elements (b,c,d) have been ordinary permuted. Not taking a into account, there are now first and last elements.

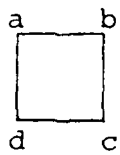
In general, $PC_m = P_{m-1} = (m-1)!$

Example. In how many different ways can seven people sit down around a circular table?

$$PC_7 = C_{7-1} = C_6 = 6! = 720 \text{ ways.}$$

Example. In how many different ways can the vertices of a square be painted with four different colors?

Recall that

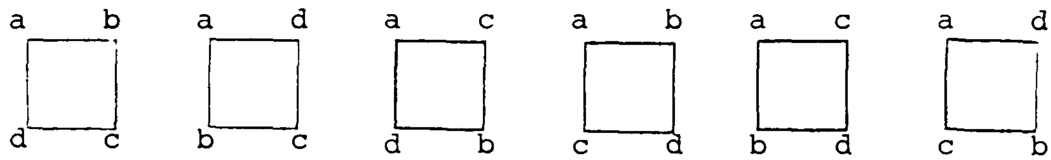


abcd
bcda
cdab
dabc

represent four different ordinary permutations (one dimension), but only one circular permutation (two dimensions).

As a square, like a circle, is a a two-dimensional figure, the solution is $PC_4 = P_3 = 3! = 3 \cdot 2 \cdot 1 = 6$

Graphically,



<u>Permutations with Repetition (Order is the Only Possible Variable)</u>. In ordinary permutations all the <u>m</u> elements belonging to the original set are assumed to be different ($a_1 \neq a_2 \neq a_3 \neq \ldots \neq a_m$).

In permutations with repetition some of those given <u>m</u> elements are repeated:

$$\overbrace{\underbrace{a_1, a_1, \ldots a_1}_{\alpha},\ \underbrace{a_2, a_2, \ldots a_2}_{\beta},\ \underbrace{a_3, a_3, \ldots a_3}_{\gamma}, \ldots,\ \underbrace{a_\ell,\ a_\ell, \ldots\ a_\ell}_{\lambda}}^{m}$$

where

element a_1 is repeated α times

element a_2 is repeated β times

element a_3 is repeated γ times

---

element a_ℓ is repeated λ times

being $\alpha + \beta + \gamma + \ldots + \lambda = m$

A procedure for calculating the general formula of permutations with repetition could be:

Suppose you have an original arrangement of those m elements

$$\overbrace{\underbrace{a_1a_1\ldots a_1}_{\alpha}\underbrace{a_2a_2\ldots a_2}_{\beta}\underbrace{a_3a_3\ldots a_3}_{\gamma}\ldots\underbrace{a_\ell a_\ell\ldots a_\ell}_{\lambda}}^{m}$$

If the α a_1 elements are switched between them, there will be $P_\alpha = \alpha!$ identical permutations.

If the β a_2 elements are switched between them, there will be $P_\beta = \beta!$ arrangements exactly equal.

The same reasoning is applicable to the rest, getting $P_\gamma = \gamma!$, ..., $P_\lambda = \lambda!$ permutations that are identical to the initial one.

Thus, the total number of permutations that end up being identical to the original arrangement is $\alpha!\beta!\gamma!\ldots\lambda!$

Therefore, dividing the ordinary permutations (P_m) of the m elements between the total number that each one of those permutations is found repeated, the result will be the number of different permutations.

The result of this process is called permutations with repetition, and is represented by

$$PR_m^{\alpha,\beta,\gamma,\ldots,\lambda} = \frac{P_m}{\alpha!\beta!\gamma!\ldots\lambda!} = \frac{m!}{\alpha!\beta!\gamma!\ldots\lambda!}$$

<u>Example</u>. In how many different ways can be ordered 20 books in a book shelf knowing that 6 have a black cover, 5 a white cover and 9 a red cover?

$$PR_{20}^{6,5,9} = \frac{20!}{6!5!9!} = 77,659,752 \text{ ways}$$

<u>Example</u>. What is the number of different permutations of the letters in the word Mississippi?

There are 1 M
4 i's
4 s's
2 p's
11 letters

Thus, $PR_{11}^{1,4,4,2} = \frac{11!}{1!4!4!2!} = 34,650$ permutations

<u>Power of a Polynomial (Leibniz' Formula)</u>.

The purpose of computing the power of a polynomial is to calculate the value of expressions like

n

$$(a + b + c + \ldots + \ell)^m$$

n

being $a \neq b \neq c \neq \ldots \neq \ell$

In general, it can be said that

$$(\overbrace{a+b+c+\ldots+\ell}^{n})^m = \Sigma\, a^{\alpha} b^{\beta} c^{\gamma} \ldots \ell^{\lambda},$$

where there are n different factors ($a \neq b \neq c \neq \ldots \neq \ell$) in every term of the sum $\Sigma a^{\alpha} b^{\beta} c^{\gamma} \ldots \ell^{\lambda}$, albeit there are a total of m factors ($\alpha+\beta+\gamma+ \ldots +\lambda$) in all of the summed terms.

The following question is how to calculate those terms with their respective coefficients.

The coefficient for every term can be obtained using permutations with repetitions, because any term

$$\overbrace{aa\ldots a}^{\alpha}\overbrace{bb\ldots b}^{\beta}\overbrace{cc\ldots c}^{\gamma}\ldots\overbrace{\ell\ell\ldots\ell}^{\lambda} = a^{\alpha} b^{\beta} c^{\gamma} \ldots \ell^{\lambda}$$

will be repeated $PR_m^{\alpha,\beta,\gamma,\ldots,\lambda} = \dfrac{m!}{\alpha!\beta!\gamma!\ldots\lambda!}$

being $\alpha+\beta+\gamma+\ldots+\lambda = m$.

Thus, the expression has turned to be

$$(\overbrace{a+b+c+\ldots+\ell}^{n})^m = \Sigma \frac{m!}{\alpha!\beta!\gamma!\ldots\lambda!} \cdot a^{\alpha} b^{\beta} c^{\gamma} \ldots \ell^{\lambda},$$

but how to know the value of $\alpha, \beta, \gamma, \ldots \lambda$? A tricky method will be taught. Splitting the number m of total factors ($m=\alpha+\beta+\gamma+\ldots+\lambda$) into all possible natural (nonfractional) subsets of n different factors, their respective exponents will be derived.

For instance, $\overbrace{(a+b+c+d)}^{4}{}^{5} = \Sigma \frac{5!}{\alpha!\beta!\gamma!\delta!} \; a^{\alpha}b^{\beta}c^{\gamma}d^{\delta}$

$\alpha+\beta+\gamma+\delta = m = 5 =$ # of <u>total</u> factors

$\overbrace{a \neq b \neq c \neq d}^{n}$; $n = 4 =$ # of <u>different</u> factors

Splitting 5 into 4 subsets, in all possible natural ways (fractions cannot be used), the values for exponents α, β, γ and [illegible] are:

$m = 5 + 0 + 0 + 0$ All [illegible]

$m = 4 + 1 + 0 + 0$ are [illegible].

$m = 3 + 2 + 0 + 0$

$m = 3 + 1 + 1 +$ [illegible]

$m = 2 + 2 + 1 +$ [illegible]

$m = 2 + 1 + 1 +$ [illegible]

Thus, $(a+b+c+d)^{5} =$ [illegible]

$\frac{5!}{5!0!0!0!0!}$ [illegible]

$+ \quad \frac{5!}{4!1!0!0!} (a^4b+a^4c+a^4d+b^4a+b^4c+b^4d+c^4a+c^4b+c^4d+d^4a+d^4b+d^4c) +$

$+ \quad \frac{5!}{3!2!0!0!} (a^3b^2+a^3c^2+a^3d^2+b^3a^2+b^3c^2+b^3d^2+c^3a^2+c^3b^2+c^3d^2+d^3a^2+d^3b^2+d^3c^2) +$

$$+ \quad \frac{5!}{3!1!1!0!}\,(a^3bc+a^3bd+a^3cd+b^3ac+b^3ad+b^3cd+c^3ab+c^3ad+$$
$$+c^3bd+d^3ab+d^3ac+d^3bc)$$

$$+ \quad \frac{5!}{2!2!1!0!}\,(a^2b^2c+a^2b^2d+a^2c^2b+a^2c^2d+a^2d^2b+a^2d^2c+b^2c^2a+$$
$$+b^2c^2d+b^2d^2a+b^2d^2c+c^2d^2a+c^2d^2b)$$

$$+ \quad \frac{5!}{2!1!1!1!}\,(a^2bcd+b^2acd+c^2abd+d^2abc)$$

In general, the Leibniz Formula for obtaining the power $\underline{m}$ of a polynomial expression composed by $\underline{n}$ different elements, is

$$(\overbrace{a+b+c+\ldots+\ell}^{n})^m = \sum \frac{m!}{\alpha!\beta!\gamma!\ldots\lambda!}\, a^\alpha b^\beta c^\gamma \ldots \ell^\lambda$$

being $\alpha+\beta+\gamma+\ldots+\lambda = m$

where $\alpha,\beta,\gamma,\ldots,\lambda$ receive all possible systems of natural values into which $\underline{m}$ can be split. NOTE: The power of a binomial (Newton's Formula), that will be seen in ordinary combinations, is a particular case of the power of a polynomial (Leibniz' Formula).

Ordinary Combinations (Order is Irrelevant). Recall that two variations are different either because (1) they differ in one element at least; or (2) they have the same

elements but in different order. In contradistinction to the case of variations where either one or both variables may be relevant, in the case of combinations there is only one variable: some element(s) are different.

Thus, given a set of $\underline{m}$ distinct objects, any unorder subset of $\underline{n}$ of the objects is called an ordinary combination. Consequently, two combinations are the same if both have the same elements, even though they are arranged differently; and two combinations are different if they differ in one element at least.

The number of combinations of size $\underline{n}$, made from a set of $\underline{m}$ different elements $(a_1 \neq a_2 \neq a_3 \neq \ldots \neq a_m)$, will be symbolized by C_m^n or $\binom{m}{n}$

Notice that here $m \geq n$.

A method for computing the general formula of ordinary combinations could be:

Suppose you already know all possible C_m^n (where order is not relevant).

Take every single combination from C_m^n and permute it (P_n) in all possible ways. For every single combination you will get P_n different permutations (where order is the unique variable).

The total number of configurations that you will have now is $C_m^n \times P_n$, because all the combinations (C_m^n) have been arranged in every conveivable manner, and that, by definition, are ordinary variations. So,

$$C_m^n \times P_n = V_m^n$$

Therefore,

$$C_m^n = \frac{V_m^n}{P_n} = \frac{\frac{m!}{(m-n)!}}{n!} = \frac{m!}{(m-n)!n!}$$

Example. How many different selections of 11 soldiers from a group of 14 can be made?

$$C_{14}^{11} = \frac{14!}{(14-11)!11!} = \frac{14!}{3!11!} = 364 \text{ selections.}$$

Example. Form the ordinary combinations of length 2 of the objects a, b, c, d, e

ab, ac, ad, ae,

bc, bd, be,

cd, ce,

de

$$10 = C_5^2 = \frac{5!}{2(5-2)!} = \frac{5!}{2!3!} = \frac{5.4}{2} = \frac{20}{2} = 10$$

Properties.

$$\binom{m}{n} = \binom{m}{m-n}$$

Proof.

$$\binom{m}{n} = C_m^n = \frac{m!}{(m-n)!n!}$$

$$\binom{m}{m-n} = C_m^{m-n} = \frac{m!}{[m-(m-n)](m-n)!} = \frac{m!}{(m-n)!n!}$$

Q.E.D

$$\binom{m}{n} = \binom{m-1}{n} + \binom{m-1}{n-1}$$

<u>Proof</u> .

$$\binom{m-1}{n} = \frac{(m-1)!}{(m-1-n)!n!} = \frac{(m-1)!}{(m-n-1)!n!}$$

$$\binom{m-1}{n-1} = \frac{(m-1)!}{[m-1-(n-1)]!(n-1)1} = \frac{(m-1)!}{(m-n)!(n-1)!}$$

$$\text{SUM} = \frac{(m-1)!}{(m-n-1)!n!} + \frac{(m-1)!}{(m-n)!(n-1)!}$$

$$= \frac{(m-1)!(m-n)}{(m-n-1)!n!(m-n)} + \frac{(m-1)!n}{(m-n)!(n-1)!n}$$

$$= \frac{(m-1)!(m-n)+(m-1)!n}{(m-n)!n!} = \frac{(m-1)![m-n+n]}{(m-n)!n!}$$

$$= \frac{(m-1)!m}{(m-n)!n!} = \frac{m!}{(m-n)!n!} = \binom{m}{n} = C_m^n$$

Q.E.D.

<u>The Tartaglia's Triangle</u>. Being $\binom{m}{n} = \binom{m-1}{n} + \binom{m-1}{n-1}$, the Tartaglia's Triangle is, by definition,

$$
\begin{array}{lccccccccc}
m = 0 & & & & & \binom{0}{0} & & & & \\
m = 1 & & & & \binom{1}{0} & & \binom{1}{1} & & & \\
m = 2 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
m = 3 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
m = 4 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

$$
\begin{array}{lccccccccccc}
m = 0 & & & & & & 1 & & & & & \\
m = 1 & & & & & 1 & & 1 & & & & \\
m = 2 & & & & 1 & & 2 & & 1 & & & \\
m = 3 & & & 1 & & 3 & & 3 & & 1 & & \\
m = 4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
m = 5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

. . . .

Power of a Binomial (Newton's Formula).

Given m binomial factors, the product is

$$
\begin{aligned}
&(x+a_1)(x+a_2)(x+a_3)\ \ldots\ (x+a_{m-1})(x+a_m) \\
&\quad = x^m + x^{m-1}(a_1+a_2+a_3+\ \ldots\ + a_m) \\
&\qquad + x^{m-2}(a_1a_2+a_1a_3+a_1a_4+\ \ldots\ + a_{m-1}a_m) \\
&\qquad + x^{m-3}(a_1a_2a_3+a_1a_2a_4+a_1a_2a_5 +\ \ldots \\
&\qquad\qquad +a_{m-2}a_{m-1}a_m) \\
&\qquad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\qquad + x(a_1a_2a_3\ \ldots\ a_{m-1}+a_1a_3a_4\ \ldots\ a_m+\ \ldots \\
&\qquad\qquad +a_2a_3\ \ldots\ a_m)+a_1a_2a_3\ \ldots\ a_{m-1}a_m
\end{aligned}
$$

If $a_1 = a_2 = a_3 = \ldots = a_m$, the new development of the product is

$$(x+a)^m$$

$$= x^m + x^{m-1} \cdot a \cdot C_m^1$$

$$+ x^{m-2} \cdot a^2 \cdot C_m^2$$

$$+ x^{m-3} \cdot a^3 \cdot C_m^3$$

$$+ x \cdot a^{m-1} \cdot C_m^{m-1}$$

$$+ a^m$$

Therefore,

$$(x+a)^m = \binom{m}{0}x^m a^0 + \binom{m}{1}x^{m-1}a + \binom{m}{2}x^{m-2}a^2$$

$$+ \binom{m}{3}x^{m-3}a^3 + \ldots + \binom{m}{m-1}xa^{m-1} + \binom{m}{m}x^0 a^m$$

<u>Example</u>.

$$(x+a)^6 = \binom{6}{0}x^0 a^6 + \binom{6}{1}xa^5 + \binom{6}{2}x^2 a^4$$

$$+ \binom{6}{3}x^3 a^3 + \binom{6}{4}x^4 a^2 + \binom{6}{5}x^5 a + \binom{6}{6}x^6 a^0$$

The coefficient for every term can be obtained using the Tartaglia's Triangle:

$$
\begin{array}{lccccccccccccc}
m = 0 & & & & & & & \binom{0}{0} & & & & & & \\
m = 1 & & & & & & \binom{1}{1} & & \binom{1}{1} & & & & & \\
m = 2 & & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
m = 3 & & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
m = 4 & & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
m = 5 & & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
m = 6 & \binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}
$$

$$
\begin{array}{lccccccccccccc}
m = 0 & & & & & & & 1 & & & & & & \\
m = 1 & & & & & & 1 & & 1 & & & & & \\
m = 2 & & & & & 1 & & 2 & & 1 & & & & \\
m = 3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
m = 4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
m = 5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
m = 6 & 6 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

$$\binom{6}{0} = 1 = \binom{6}{6}$$

$$\binom{6}{1} = 6 = \binom{6}{5}$$

$$\binom{6}{2} = 15 = \binom{6}{4}$$

$$\binom{6}{3} = 20 = \binom{6}{3}$$

Therefore,

$$(x+a)^6 = a^6 + 6xa^5 + 15x^2a^4 + 20x^3a^3 + 15x^4a^2 + 6x^5a + x^6$$

Note the symmetry of the coefficients (1-6-15-20-15-6-1).

The same result should be obtained usin.' the Liebniz Formula for the power of a multinomial.

$$(x+a)^6 = \frac{6!}{\alpha!\beta!} \Sigma \; x^\alpha a^\beta = PR_6^{\alpha,\beta} \cdot \Sigma \; x^\alpha a^\beta$$

$\alpha + \beta = m = 6$; $n = 2$

Set of natural values for α and β

$m = 6 + 0$

$m = 5 + 1$

$m = 4 + 2$

$m = 3 + 3$

Thus, $(x+a)^6$

$$= \frac{6!}{6!0!} (x^6+a^6) + \frac{6!}{5!1!}(x^5a+xa^5)$$

$$+ \frac{6!}{4!2!} (x^4a^2+x^2a^4) + \frac{6!}{3!3!} \cdot x^3a^3$$

$$= x^6 + a^6 + 6x^5a + 6xa^5 + 15x^4a^2 + 15x^2a^4 + 20x^3a^3$$

$$= a^6 + 6xa^5 + 15x^2a^4 + 20x^3a^3 + 15x^4a^2 + 6x^5a + x^6$$

Q.E.D.

Note again the symmetrical disposition of the coefficients (1-6-15-20-15-6-1).

Comparing both procedures, the binomial and the multinomial, a conclusion can be made that,

$$C_6^0 = C_6^6 = PR_6^{6,0} = 1$$

$$C_6^1 = C_6^5 = PR_6^{5,1} = 6$$

$$C_6^2 = C_6^4 = PR_6^{4,2} = 15$$

$$C_6^3 = C_6^3 = PR_6^{3,3} = 20$$

Note: These equalities happen because

$$C_m^n = C_m^{m-n} = PR_m^{n,(m-n)} = \frac{m!}{n!\,(m-n)!}\ ,$$

but such particular mathematical truism <u>cannot</u> be generalized in the sense that any ordinary combination can be represented by a permutation with repetition. Remember the different initial conditions: in ordinary combinations the <u>m</u> given elements are all distinct ($a_1 \neq a_2 \neq a_3 \neq \ldots \neq a_m$), but in permutations with repetition some of the m elements are repeated

$$\overbrace{\underbrace{a_1 a_1 \ldots a_1}_{\alpha}\underbrace{a_2 a_2 \ldots a_2}_{\beta}\underbrace{a_3 a_3 \ldots a_3}_{\gamma} \ldots \underbrace{a_\ell a_\ell \ldots a_\ell}_{\lambda}}^{m}$$

Example. Calculate $(a+b+c)^4$ using the Leibniz Formula.

Solution.

$$\left.\begin{array}{l} m = 4 = 4 + 0 + 0 \\ m = 4 = 3 + 1 = 0 \\ m = 4 = 2 + 2 + 0 \\ m = 4 = 2 + 1 + 1 \end{array}\right| \begin{array}{l} \text{Set of all possible values} \\ \text{for } \alpha,\ \beta \text{ and } \gamma. \\ n = 3 \\ \ \end{array}$$

$$(a+b+c)^4 = \Sigma \frac{4!}{\alpha!\beta!\gamma!} a^{\alpha}b^{\beta}c^{\gamma}$$

$$= \frac{4!}{4!0!0!} \Sigma a^4b^0c^0 + \frac{4!}{3!1!0!} \Sigma a^3b^1c^0$$

$$+ \frac{4!}{2!2!0!} \Sigma a^2b^2c^0 + \frac{4!}{2!1!1!} \Sigma a^2b^1c^1$$

$$= a^4 + b^4 + c^4 + 4(a^3b+a^3c+ab^3+b^3c+ac^3+bc^3)$$

$$+ 6(a^2b^2+a^2c^2+b^2c^2) + 12(a^2bc+ab^2c+abc^2)$$

Example. Calculate $(a+b+c)^4$ using Newton's Formula.

Solution.

$$(a+b+c)^4 = [(a+b)+c]^4$$

$$= \binom{4}{0}(a+b)^4c^0 + \binom{4}{1}(a+b)^3c^1 + \binom{4}{2}(a+b)^2c^2$$

$$+ \binom{4}{3}(a+b)^1c^3 + \binom{4}{4}(a+b)^0c^4$$

$$= (a+b)^4 + 4(a+b)^3c + 6(a+b)^2c^2 + 4(a+b)c^3+c^4$$

$$= \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}a^0b^4$$

$$+ 4\binom{3}{0}a^3b^0c + 4\binom{3}{1}a^2b^1c + 4\binom{3}{2}ab^2c + 4\binom{3}{3}a^0b^3c$$

$$+ 6\binom{2}{0}a^2b^0c^2 + 6\binom{2}{1}a^1b^1c^2 + 6\binom{2}{2}a^0b^2c^2$$

$$+ 4ac^3 + 4bc^3 + c^4$$

$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$+ 4a^3c + 12a^2bc + 12ab^2c + 4b^3c$$

$$+ 6a^2c^2 + 12abc^2 + 6b^2c^2$$

$$+ 4ac^3 + 4bc^3 + c^4$$

$$= a^4+b^4+c^4 + 4(a^3b+ab^3+a^3c+b^3c+ac^3+bc^3)$$

$$+ 6(a^2b^2+a^2c^2+b^2c^2) + 12(a^2bc+ab^2c+abc^2)$$

$$= a^4 + b^4 + c^4 + 4(a^3b+a^3c+ab^3+b^3c+ac^3+bc^3)$$

$$+ 6(a^2b^2+a^2c^2+b^2c^2) + 12(a^2bc+ab^2c+abc^2)$$

Combinations with Repeated Selection (Order is Irrelevant). If in ordinary combinations every combination is composed by different elements, in combinations with repeated selection it is possible that any element can be repeated any number of times. Therefore, combinations with repeated selection are different subsets of $\underline{n}$ elements each, which have been taken from a given set of $\underline{m}$ different elements, but with the possibility for every single object to go back and forth from the given set into any subset. In this way, there are always $\underline{m}$ elements available for constructing subsets, as every element can be selected repeatedly again and again.

A form to represent combinations with repeated selection is CR_m^n.

Notice then here $m \gtrless n$, because there are $\underline{m}$ distinct objects ($a_1 \neq a_2 \neq a_3 \neq \ldots \neq a_m$) in the given set, each of them can be indefinitely used for constructing subsets of length $\underline{n}$.

The general formula for combinations with repeated selection can be deduced using an auxiliary transformation. Suppose the combinations with repeated selection are all known and perform the following auxiliary transformation to every combination:

$$\begin{array}{lcl}
\overbrace{a_1 a_1 a_1 a_2 a_3 \cdots a_{k<m}}^{n} & \xrightarrow{\text{auxiliary transformation}} & b_1 b_{1+1} b_{1+2} b_{2+3} b_{3+4} \cdots \overbrace{b_{k+(n-1)}}^{n} \\
a_3 a_4 a_4 a_7 a_{11} \cdots a_{p<m} & \longrightarrow & b_3 b_{4+1} b_{4+2} b_{7+3} b_{11+4} \cdots b_{p+(n-1)} \\
\text{Group [1]} & \cdots\cdots\cdots & \text{Group [2]} \\
a_1 a_2 a_3 a_5 a_5 \cdots a_m & \longrightarrow & b_1 b_{2+1} b_{3+2} b_{5+3} b_{5+4} \cdots b_{m+(n-1)}
\end{array}$$

Using the former algorithm, you have passed to the second set of elements $\underline{b}$ (Group 2) by adding consecutive natural numbers to the subindex of elements $\underline{a}$ (Group 1):

0 to the $\underline{a}$ element in the first position of each row

1 to the $\underline{a}$ element in the second position of each row

2 to the $\underline{a}$ element in the third position of each row

---

(n-1) to the a element in the n^{th} position of each row

$$\begin{array}{lcl}
\overbrace{a_1 a_1 a_1 a_2 a_3 \cdots a_{k<m}}^{n} & \longrightarrow & \overbrace{b_1 b_2 b_3 b_5 b_7 \cdots b_{k+(n-1)}}^{n} \\
a_3 a_4 a_4 a_7 a_{11} \cdots a_{p<m} & \longrightarrow & b_3 b_5 b_6 b_{10} b_{15} \cdots b_{p+(n-1)} \\
\text{Group [1]} & \cdots\cdots\cdots & \text{Group [2]} \\
a_1 a_2 a_3 a_5 a_5 \cdots a_m & \longrightarrow & b_1 b_3 b_5 b_8 b_9 \cdots b_{m+(n-1)}
\end{array}$$

Passing from Group [1] to Group [2] was performed using an algorithm that allows one to see a mutual correspondence between every pair in both groups: to each combination on the left side (with combination repeated selection) corresponds a combination on the right side (ordinary combination). Checking the new subindex for elements in the right side group, the highest subindex of b must correspond to $a_m \longrightarrow b_{m+(n-1)}$ because m is the highest subindex in group a and it is allocated the last one; so the algorithm gives to it the biggest subindex (m+n-1) in group b.

Therefore, a general formula can be established

$$CR_m^n = C_{m+n-1}^n = \binom{m+n-1}{n}$$

Example. How many different subsets of two coins can be made with pennies, nickels, dimes and quarters?

$$CR_4^2 = C_{4+2-1}^2 = C_5^2 = \frac{5!}{(5-2)!2!} = 10 \text{ subsets}$$

Section II. Problems

Introduction. This section will present the solution of concrete problem scenarios in one and two dimensional space using the framework provided by the conceptual map. The conceptual map is updated in Figure 3 which includes the general formulas for main and subordinate

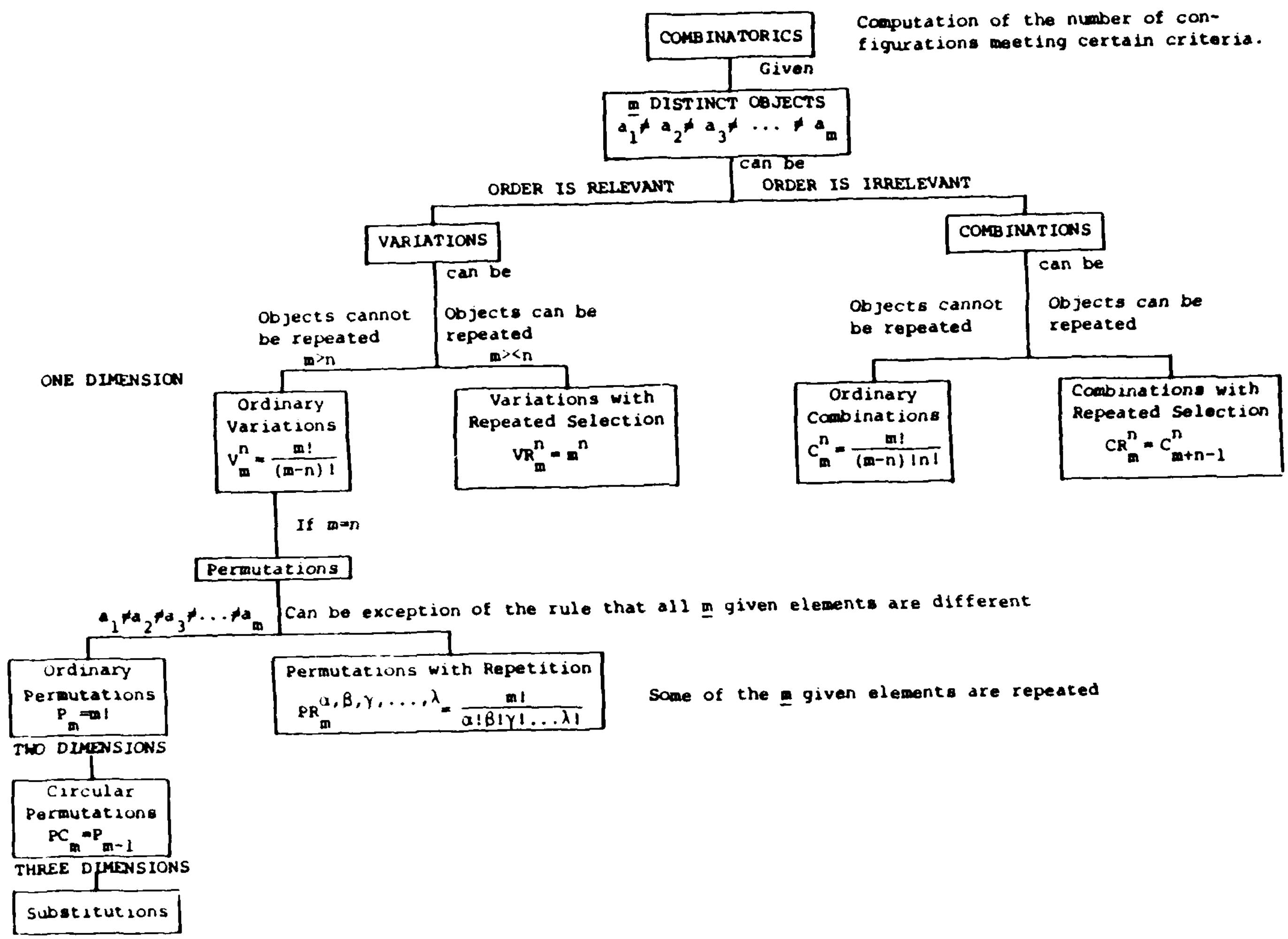


Fig. 3. Conceptual Map

concepts already obtained. Substitutions will be explained in the third section along with some problems in three dimensions.

<u>Problem No. 1. Ordinary Variations</u>. "In how many ways can 3 prizes be distributed to twenty competitors if each person can receive at most 1 prize" (Eisen, 1969:12)?

<u>Solution</u>. Assume you have been asked to assign people (all different, of course) to three prizes (different, too). This is the most general and inclusive concept in combinatorics (variations), where order is relevant.

Thus,

$$V_{20}^{3} = \frac{20!}{(20-3)!} = \frac{20!}{17!} = 20 \cdot 19 \cdot 18 = 6840 \text{ ways}$$

<u>Problem No. 2. Ordinary Variations</u>. "In how many ways can four letters be put in four envelopes, one in each" (Eisen, 1969:8)?

<u>Solution</u>. Order is relevant.

$$V_{4}^{4} = \frac{4!}{(4-4)!} = \frac{4!}{0!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{1} = 24 \text{ ways.}$$

Note that this problem can be solved using ordinary permutations because permutations are a particular case of variations when m = n.

Thus,

$$V_m^m = P_m = m!$$

$$V_4^4 = P_4 = 4! = 24$$

Problem No. 3. Ordinary Variations. "How many different numbers can be formed using the digits 1, 2, 3, 4 if repetitions of the digits are not allowed" (Eisen, 1969:10)?

Solution. Order is relevant. Length of the numbers (the value of *n*) is relevant.

Thus,

$$V_4^1 + V_4^2 + V_4^3 + V_4^4$$

$$= \frac{4!}{(4-1)!} + \frac{4!}{(4-2)!} + \frac{4!}{(4-3)!} + \frac{4!}{(4-4)!}$$

$$= 4 + 12 + 24 + 24 = 64 \text{ different numbers.}$$

Problem No. 4. Ordinary Variations. "How many different 3 letter words (with no repetition) can be formed from the letters a, b, c, d, e, f, g, h which (1) include the letter *e*; and (2) do not include the letter *e*" (Eisen, 1969:13)?

Solution.

1. There are a total of 8 different letters. Letter e can be fixed in three different places:

$\underline{e}$ _ _

_ $\underline{e}$ _

_ _ $\underline{e}$

Order is relevant here, and repetition is not allowed. Thus, the other seven letters can be chosen in V_7^2 ways. So,

3 places for $\underline{e}$ x V_7^2 ways for the other 7 letters

$$= 3 \cdot \frac{7!}{(7-2)!} = 3 \cdot \frac{7!}{5!} = 3 \cdot 7 \cdot 6 = \underline{126 \text{ words}}$$

2. Excluding $\underline{e}$, there are only 7 letters, that make

$$V_7^3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 7 \cdot 6 \cdot 5 = \underline{210 \text{ words}}.$$

Problem No. 5. Ordinary Variations. "How many distinct 3 digit numbers are there which are even and have no repeated digits" (Eisen, 1969:16)?

Solution.

___ ___ $\underset{0,2,4,\ 6,8}{\underline{\quad}}$ The last digit has to be 0, 2, 4, 6, or 8

If last digit is zero ___ ___ 0 V_9^2

If last digit is 2, 4, 6, or 8 ___ ___ $\underset{2,4,\ 6,8}{\underline{\quad}}$

$4 \times (V_9^2 - V_8^1)$

Numbers beginning with 0 are not three digit numbers and have to be subtracted.

Thus,

$$V_9^2 + 4(V_9^2 - V_8^1) = \frac{9!}{(9-2)!} + 4\ (\frac{9!}{(9-2)!} - \frac{8!}{(8-1)!})$$

$$= \frac{9!}{7!} + 4\ (\frac{9!}{7!}) - \frac{8!}{7!})$$

$$= 9 \cdot 8 + 4(9 \cdot 8 - 8) = 9 \cdot 8 + 4 \cdot 8 \cdot (9-1)$$

$$= 9 \cdot 8 + 4 \cdot 8 \cdot 8 = 72 + 256 = \underline{328 \text{ numbers}}.$$

Problem No. 6. Variations with Repeated Selection.

How many numbers of five digits are in the base 4 numerical system?

Solution. Digits available in base 4 are four: 0, 1, 2, and 3. Order is relevant. Numbers can be repeated. Numbers beginning with 0 are not five digit numbers, and they have to be subtracted.

Thus,

$$VR_4^5 - VR_4^4 = 4^5 - 4^4 = 4^4(4-1)$$

$$= 4^4 \cdot 3 = 256 \cdot 3 = \underline{768 \text{ numbers}}.$$

Problem No. 7. Variations with Repeated Selection.

(1) How many reversible (symmetrical) numbers with six digits exist in our decimal numerical system? and (2) Add them.

Solution.

1. Symmetrical structure $\boxed{a|b|c|c|b|a}$

Order is relevant. Working with $\boxed{a|b|c}$ is enough $\longrightarrow$ VR_{10}^{3}. But those numbers starting with 0 are not six digit numbers; and, therefore, they have to be subtracted.

Thus,

$$\overbrace{\boxed{0|b|c}}^{2} \quad VR_{10}^{3} - VR_{10}^{2} = 10^{3} - 10^{2} = 10^{2}\ (10-1)$$

$$= 9 \cdot 10^{2} = \underline{900 \text{ numbers}}.$$

2. In order to add these 900 numbers, three steps are required:

a. Lateral or peripheral columns. There are only numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 (no 0's). How many? Each number is equally distributed 900/9 = 100 times. Thus, each number has to be added 100 times,

$$100 \times (1+2+3+4+5+6+7+8+9) = 100 \times 45 = \underline{4,500}.$$

b. Intermediate columns. Number 0 appears here. So, there are 900/10 = 90 times that each number 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 has to be added.

Thus,

$$90 \times (0+1+2+3+4+5+6+7+8+9) = 90 \times 45 = \underline{4050}.$$

c. $a\ b\ c\ c\ b\ a = a \times 10^{5} + b \times 10^{4}$

$$+ c \times 10^{3} + c \times 10^{2} + b \times 10 + a$$

$$
\begin{array}{lr}
= & 450000000 \\
 & 40500000 \\
 & 4050000 \\
+ & 405000 \\
 & 40500 \\
 & 4500 \\
\hline
 & 495{,}000{,}000
\end{array}
$$
symmetrical numbers

Problem No. 8. Variations with Repeated Selection.

"How many different outcomes are possible when 100 different dice are rolled" (Eisen, 1979:9)?

Solution. Every dice has 6 different faces. The order of presentation of any of such six faces, after every dice is rolled, is relevant. Each value is repeated 100 times. Thus,

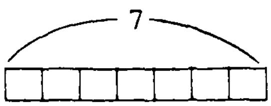
$$VR_6^{100} = 6^{100} \text{ different outcomes.}$$

Problem No. 9. Variations with Repeated Selection.

1. How many numbers with seven digits exist in our numerical system?

2. How many of those numbers have four 2's and three 5's? Add them.

Solution.

1. 

Order is relevant.

When calculating VR_{10}^{7}, all possible seven digit numbers are obtained, including those beginning with 0, which are not real seven digit numbers; so, they have to be subtracted.

Thus,

$$\boxed{0}\,\overbrace{\_\ \_\ \_\ \_\ \_\ \_}^{6}$$

$$VR_{10}^{7} - VR_{10}^{6} = 10^{7} - 10^{6} = 10^{6}\ (10-1)$$

$$= \underline{9{,}000{,}000 \text{ seven digit numbers}}.$$

2. $$\overbrace{\underbrace{2\ 2\ 2\ 2}_{\alpha=4}\ \underbrace{5\ 5\ 5}_{\beta=3}}^{m=7}$$

$$PR_{7}^{4,3} = \frac{7!}{4!3!} = \underline{35 \text{ numbers}}$$

For adding those 35 numbers consider that there are a total of 35 rows to be added; that each row is composed of seven elements of which four are 2's and three are 5's.

$$\text{Structure} \qquad \begin{array}{c} \text{7 columns} \\ \begin{array}{ccccccc} 2 & 2 & 2 & 2 & 5 & 5 & 5 \\ 2 & 5 & 2 & 5 & 2 & 5 & 2 \\ 2 & 2 & 5 & 5 & 5 & 2 & 2 \\ 2 & 5 & 5 & 2 & 2 & 2 & 5 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \end{array} \quad \text{35 rows}$$

In every column there are a total of 35 numbers; these numbers are just 2's and 5's in amounts proportionally distributed to 4 and 3 respectively. Dividing 35 proportionally to 4 and 3,

$$\frac{\#\text{ of 2's}}{4} = \frac{\#\text{ of 5's}}{3} = \frac{\#\text{ of 2's} + \#\text{ of 5's}}{4+3} = \frac{35}{7}\ ;$$

$$\# \text{ of } 2\text{'s} = 4 \cdot \frac{35}{7} = 20 \text{ twos.}$$

$$\# \text{ of } 5\text{'s} = 3 \cdot \frac{35}{7} = 15 \text{ fives.}$$

Adding the 35 numbers of any column (without carrying),

$$20 \text{ twos} + 15 \text{ fives} = 20 \times 2 + 15 \times 5 = 40 + 75$$
$$= 115$$

The total addition will be

$$115 + 115 \cdot 10 + 115 \cdot 10^2 + 115 \cdot 10^4 + 115 \cdot 10^5$$
$$+ 115 \cdot 10^6$$
$$= 115\,(1 + 10 + 10^2 + 10^3 + 10^4 + 10^5 + 10^6)$$
$$= \underline{127{,}777{,}765}$$

Problem No. 10. Variations with Repeated Selection. "Among the 10 billion numbers between 1 and 10,000,000,000, how many of them contain the digit 1? How many of them do not" (Liu, 1968:6)?

Solution. Calculate how many numbers do not contain the digit 1.

0 1 2 3 4 5 6 7 8 9

Among 0 and 9,999,999,999 there are $VR_9^{10} = 9^{10}$ numbers that do not contain the digit 1. Think that, as repetition is allowed, number 0 plays a basic role when it

is conveniently positioned in first place in order to construct all numbers of length < 10.

Among 1 and 10,000,000,000 there will be $VR_9^{10} - 1$ numbers that do <u>not</u> contain digit 1, because now number 0,000,000,000 is excluded, but also because the new 10,000,000,000 does not count here either, because this number contains digit 1.

Thus, there are $VR_9^{10} - 1 = 9^{10} - 1$ numbers that do not contain the digit 1.

The total number of numbers containing the digit 1 is calculated by the difference

$$10^{10} - (VR_9^{10} - 1) = 10^{10} - (9^{10} - 1)$$

$$= 10^{10} - 9^{10} + 1 = \underline{6,513,215,600}$$

<u>Problem No. 11. Ordinary Permutations</u>. How many words can be formed with $\underline{n}$ different vowels and $\underline{n}$ different consonants in such a way that there are not two vowels, and not two consonants together?

<u>Solution</u>.

____.____.____.

.____.____.____   Order is relevant.

Suppose every line corresponds to a vowel and every dot to a consonant.

Permuting vowels: P_n

Permuting consonants: P_n

Every word can begin either with a vowel or consonant.

Thus,

$$P_n \times P_n \times 2 = 2 \cdot (P_n)^2$$

Problem No. 12. Ordinary Permutations. "In how many ways can 10 men be arranged in a row given that three particular men must always stand next to each other" (Eisen, 1969:16)?

Solution.

$$\underline{1}\ \underline{2}\ \underline{3}\ \underline{4}\ \underbrace{\boxed{5\ 6\ 7}}_{\text{fixed}}\ \underline{8}\ \underline{9}\ \underline{10}$$

Consider, for example, the block made by the three elements 5, 6, 7 . This block can be internally rearranged in $P_3 = 3! = 6$ ways.

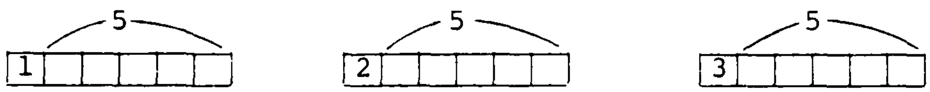
Considering this block 5, 6, 7 as one single element, the whole group is now composed by 8 elements, which can be permuted in $P_8 = 8! = 40,320$ ways.

Thus, the result is

$$P_3 \times P_8 = 6 \times 40,320 \approx \underline{241,920 \text{ ways}}.$$

Problem No. 13. Ordinary Permutations. If the numbers obtained by permuting 1, 2, 3, 4, 7 and 9 are put in increasing order of value, what position will the number 432917 occupy?

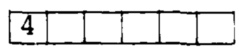
Solution.



Quantity of numbers starting with 1: $P_5 = 5! = 120$

Quantity of numbers starting with 2: $P_5 = 5! = 120$ Total 360

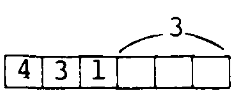
Quantity of numbers starting with 3: $P_5 = 5! = 120$

The total number of numbers starting with 1, 2, and 3 is 360. Now, number 4 is in the first place 

Quantity of numbers with 1 in second place: $P_4 = 4! = 24$ Total

Quantity of numbers with 2 in second place: $P_4 = 4! = 24$ 48

The total number of numbers starting with 41 and 42 is 48. Now 43 is in first places.



Quantity of numbers with 1 in third place: $P_3 = 3! = 6$ Total 6

Now 432 are the three first digits.

$$\boxed{4}\boxed{3}\boxed{2}\overbrace{\boxed{}\boxed{}\boxed{}\boxed{}}^{3} \qquad \boxed{4}\boxed{3}\boxed{2}\boxed{1}\overbrace{\boxed{}\boxed{}}^{2} \qquad \boxed{4}\boxed{3}\boxed{2}\boxed{7}\overbrace{\boxed{}\boxed{}}^{2}$$

Quantity of numbers with 1 in fourth place: $P_2 = 2$ Total

Quantity of numbers with 7 in fourth place: $P_2 = 2$ 4

The total number of numbers starting with 432 is 4. Now 4329 are the four first digits.

$$\boxed{4}\boxed{3}\boxed{2}\boxed{9}\boxed{}\boxed{}\boxed{} \qquad \boxed{4}\boxed{3}\boxed{2}\boxed{9}\boxed{1}\overbrace{\boxed{}}^{1}$$

Quantity of numbers with 1 in fifth position: $P_1 = 1$ Total 1

The last square has to be assigned to number 7.

Thus, the solution is

$$360 + 48 + 6 + 4 + 1 = \underline{419^{th} \text{ place}}.$$

<u>Problem No. 14. Permutations with Repetition</u>.

"How many different words can be formed by rearranging the letters of the word ENGINEER" (Eisen, 1969:14)?

<u>Solution</u>. There are 3 E's

2 N's

and a total of 8 letters.

Thus, $PR_8^{3,2} = \frac{5!}{3!2!} = \frac{5 \cdot 4}{2} = \underline{10 \text{ words}}.$

Problem No. 15. Permutations with Repetition.

What is the number of ways of painting 7 different cars knowing that 2 cars have to be painted in white, 4 in black and 1 in green?

Solution.

$$PR_7^{2,4,1} = \frac{7!}{2!4!1!} = \underline{105 \text{ ways}}.$$

Problem No. 16. Permutations with Repetition.

"In how many ways can three 6's and two 5's be obtained when five dice are cast" (Eisen, 1969:16)?

Solution.

If three 6's and two 5's have always appeared (according to the data), the only way to differentiate between different results is permuting the order of the outcomes and eliminating duplication.

Thus,

$$PR_5^{3,2} = \frac{5!}{3!2!} = \frac{5 \cdot 4}{2} = \underline{10 \text{ ways}}.$$

Problem No. 16. Permutations with Repetition.

"Five distinct letters are to be transmitted through a communications channel. A total of 15 blanks are to be inserted between the letters with at least three blanks between every two letters. In how many ways can the letters and blanks be arranged" (Liu, 1968:14)?

Solution.

space space space space

a $\boxed{}$ b $\boxed{}$ c $\boxed{}$ d $\boxed{}$ e

There are $P_5 = 5!$ ways to arrange the letters. The number of blanks inserted between every two letters have to be at least three. One can insert three blanks (the required minimum amount) in the open spaces, which would make

4 spaces x 3 blanks = 12 blanks

15 blanks - 12 blanks = 3 remaining blanks.

The problem can be readdressed now under the following terms: in how many different ways can three non-distinct objects be distributed into four boxes?

1	1	1	0	$PR_4^3 = \frac{4!}{3!} = 4$
2	1	0	0	$PR_4^2 = \frac{4!}{2!} = 12$
3	0	0	0	$PR_4^3 = \frac{4!}{3!} = 4$

The result is

$$P_5 \ (4 + 12 + 4)$$

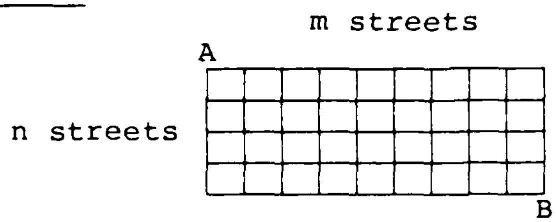
$$= 5! \times 20 = 120 \times 20 = \underline{2{,}400 \text{ ways}}.$$

Problem No. 18. Permutations with Repetition.

A city has rectangular shape and it is crossed by m streets going North-South, and by n streets going East-West, including peripheral streets.

In how many ways can we pass from a corner A to its opposite vertex B without going backwards anytime?

Solution.



For going from A to B, you have to travel, whatever route you choose, a total amount of (m-1) + (n-1) blocks = m + n - 2 constant blocks, of which (m-1) blocks have to be traveled vertically (v), and (n-1) horizontally (h).

Those (m-1) blocks in vertical direction have equal opportunity of selection in the whole city; and those (n-1) blocks in horizontal direction have an equal chance for being chosen too.

Thus, a permutation like

v v h v h h ... v

means that you have to travel two blocks vertically $\downarrow\downarrow$, then one block horizontally $\longrightarrow$, later one block vertically, then two blocks horizontally, etc.

So, the solution will be

$$PR^{(m-1),(n-2)}_{(m+2-2)} = \frac{(m+n-2)!}{(m-1)!(n-2)!}$$

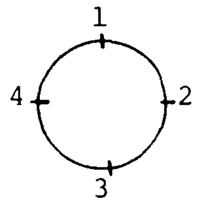
<u>Problem No. 19. Circular Permutations</u>. In how many different ways can four people sit down around a circular table?

<u>Solution</u>.

$$PC_4 = P_{4-1} = P_3 = 3! = 3 \cdot 2 \cdot 1 = \underline{6\ \text{ways}}.$$

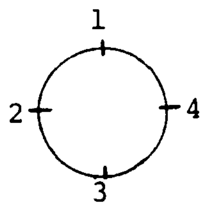
<u>Problem No. 20. Circular Permutations</u>. "In how many different ways can four different colored spherical beads be strung on a string to form a necklace" (Eisen, 1969:17)?

<u>Solution</u>. Recall that in circular permutations $PR_m = P_{m-1}$.
$PC_4 = P_{4-1} = P_3 = 3! = 6$.



[1]

Element 1 has been fixed, for instance, and the other three beads have been permuted around it. One of these permutations is, for example,



[2]

Because the necklace can be taken in hand and flipped 180° in the space, figures [1] and [2] are the same. Notice that, under this consideration, the problem is three dimensional. Thus, the result is

$$\frac{PC_4}{2} = \frac{6}{2} = \underline{3\text{ different ways}}.$$

Problem No. 21. Ordinary Combinations. In how many different ways can we put in a row p positive signs (+) and n negative signs (-), being n < p, in such a manner that two negative signs should not be together?

Solution.

$$\begin{array}{ccccccccccccccc} . & + & . & + & . & + & . & + & \text{--------} & . & + & . & + & . \\ 1 & & 2 & & 3 & & 4 & & & p\text{-}1 & & p & & p+1 \end{array}$$

As you can see in the above helping figure, there are (p+1) possible places available for placing the n negative signs in. Order is not relevant. Thus,

$$C^n_{p+1} = \binom{p+1}{n}$$

Problem No. 22. Ordinary Combinations. "In how many ways can a committee of 3 teachers and 4 students be chosen from 9 teachers and 6 students knowing that teacher A refuses to serve if student B is a member" (Eisen, 1969:23)?

Solution.

Ways to select teachers: C_9^3. Order is irrelevant.

Ways to select students: C_6^4. Order is irrelevant.

Total number of ways to select teachers and students: $C_9^3 \times C_6^4$.

But because of the given constraint, those combinations where A and B appear together have to be deleted, such as

teachers	students
(A ___ ___)	(B ___ ___ ___)

Eliminating teacher A, ways to select 2 teachers: C_8^2.

Eliminating student B, ways to select 3 students: C_5^3.

Total number of combinations to be deleted: $C_8^2 \times C_5^3$.

Thus, the result will be:

$$C_9^3 \times C_6^4 - C_8^2 \times C_5^3$$

$$= \frac{9!}{6!3!} \cdot \frac{6!}{2!4!} - \frac{8!}{6!2!} \cdot \frac{5!}{3!2!}$$

$$= 1260 - 280 = \underline{980 \text{ committees}}.$$

Problem No. 23. Ordinary Combinations. "In how many ways can three numbers be selected from the numbers 1, 2, 3, ..., 300 such that their sum is divisible by 3" (Liu, 1968:9)?

Solution. The 300 different numbers pertaining to the set (1, 2, 3, ..., 300) can be classified in three different subsets:

Subset 1, that groups all the numbers that are divisible by 3. That means that any number N_1 belonging to subset 1 makes $N_1 = \dot{3}$. Thus, any three numbers selected from subset 1 will make $n_{11} + n_{12} + n_{13} = \dot{3} + \dot{3} + \dot{3} = \dot{3}$.

Subset 2, that groups all the numbers that yield 1 as remainder when divided by 3. That means that any number N_2 belonging to subset 2 makes $N_2 = \dot{3} + 1$. Thus, any three numbers selected from subset 2 will make:

$$n_{21} + n_{22} + 2_{23}$$

$$= (\dot{3}+1) + (\dot{3}+1) + (\dot{3}+1)$$

$$= \dot{3} + 3 = \dot{3}$$

Subset 3, that groups all the numbers that yield 2 when they are divided into 3. That means that any N_3: $\dot{3} + 2$; and summing any three of those numbers belonging to subset 3,

$$n_{31} + n_{32} + n_{33} = (\dot{3}+2) + (\dot{3}+2) + (\dot{3}+2)$$

$$= \dot{3} + 6 = \dot{3}$$

It is obvious that there are $\frac{300}{3} = 100$ distinct numbers in every subset. It is evident too that one number can be selected from each subset, sum the three numbers, and the result will be divisible by three also.

$$N_1 + N_2 + N_3 = \dot{3}$$

because

$$\dot{3} + (\dot{3}+1) + (\dot{3}+2) = \dot{3} + 3 = \dot{3}$$

Numbers can be selected using ordinary combinations, where order is not important.

Thus,

3 numbers from subset 1 ⟶ $C_{100}^{3} = \frac{100!}{3!97!} = 161,700$

3 numbers from subset 2 ⟶ $C_{100}^{3} = \qquad = 161,700$

3 numbers from subset 3 ⟶ $C_{100}^{3} = \qquad = 161,700$

1 number from each subset ⟶ $C_{100}^{1} \times C_{100}^{1} \times C_{100}^{1}$

$$= (C_{100}^{1})^{3} = 1,000,000$$

Total different ways:

$$C_{100}^{3} + C_{100}^{3} + C_{100}^{3} + (C_{100}^{1})^{3}$$

$$= 3 \times 161.700 + 1,000,000 = \underline{1,485,100}.$$

<u>Problem No. 24. Combinations with Repeated Selection</u>. "Out of a large number of pennies, nickels, dimes and quarters, in how many ways can six coins be selected" (Liu, 1968:10)?

<u>Solution</u>. Order is not relevant. Elements can be taken repeatedly.

Thus,

$$CR_{4}^{6} = C_{4+6-1}^{6} = C_{9}^{6} = \frac{9!}{6!3!} = \underline{84 \text{ ways}}.$$

<u>Problem No. 25. Combinations with Repeated Selection</u>. "If a candy factory manufactures 10 different kinds

of chocolate and puts them in boxes of 32, how many different boxes can be formed" (Eisen, 1969:25)?

Solution.

$$CR_{10}^{32} = C_{10+32-1}^{32} = C_{41}^{32} = \frac{41!}{32!9!}$$

$$= \underline{350,343,570 \text{ different boxes}}.$$

Problem No. 26. Combinations with Repeated Selection. "How many divisors does the number 1400 have" (Liu, 1968:10)?

Solution. Number 1400 will be represented as a product of its prime factors. The divisors are combinations made with those prime factors.

$$1400 = 2^3 \cdot 5^2 \cdot 7$$

Ways to select factor 2 (maximum three times):

$$CR_1^0 + CR_1^1 + CR_1^2 + CR_1^3 = C_{1+0-1}^0 + C_{1+1-1}^1$$

$$+ C_{1+2-1}^2 + C_{1+3-1}^3$$

$$= C_0^0 + C_1^1 + C_2^2 + C_3^3 = 1 + 1 + 1 + 1 = 4$$

Ways to select factor 5 (maximum two times):

$$CR_1^0 + CR_1^1 + CR_1^2 = C_0^0 + C_1^1 + C_2^2 = 3$$

Ways to select factor 7 (one time only):

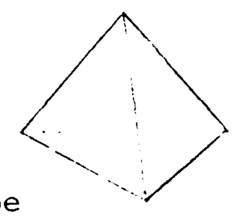
$$CR_1^0 + CR_1^1 = 1 + 1 = 2$$

Notice that factor 7 can be selected two times because $7^0 = 1$ and $7^1 = 7$ are both considered. Thus, the result will be,

$$4 \cdot 3 \cdot 2 = \underline{24 \text{ divisors}}.$$

Problem No. 27. Combinations with Repeated Selection. In how many different ways can a regular tetrahedron be painted with seven different colors such that each face is painted with one and only one color?

Solution. A tetrahedron has four faces:



On a first approach, the solution looks to be

$$CR_7^4 = C_{7+4-1}^4 = C_{10}^4 = \frac{10!}{4!(10-4)!}$$

$$= \frac{10!}{4!6!} = \underline{210 \text{ ways}}. \qquad [1]$$

Solving the problem in four analytical steps:

Step 1: Painting with 1 color, there are

$$C_7^1 = \frac{7!}{1!6!} = 7 \text{ colors for painting} \longrightarrow$$

$\longrightarrow$ 7 different ways to paint the tetrahedron.

Step 2: Painting with 2 colors, there are

$$C_7^2 = \frac{7!}{2!5!} = 21 \text{ pairs of colors for painting}$$

With those 21 pairs, the tetrahedron can be painted, for example,

1 face red, 3 faces blue 2 faces red, 2 faces blue 3 faces red, 1 face blue	There are 3 different possibilities for every pair

Thus,

21 pairs x 3 possibilities each

= 63 different ways to paint the tetrahedron.

Step 3: Painting with 3 colors, there are

$$C_7^3 = \frac{7!}{3!4!} = 35 \text{ trios of colors for painting}$$

The four faces can be painted, for instance,

1 face red, 1 face blue, 2 faces green 1 face red, 2 faces blue, 1 face green 2 faces red, 1 face blue, 1 face green	There are 3 different possibilities for every trio.

Thus,

35 trios x 3 possibilities each

= 105 different ways to paint the tetrahedron.

Step 4: Painting with 4 colors, there are

$$C_7^4 = \frac{7!}{4!3!} = 35 \text{ groups of four colors available}$$

for painting the tetrahedron $\longrightarrow$

$\longrightarrow$ 35 different ways to paint it.

Adding partial results,

$7 + 63 + 105 + 35 = \underline{210 \text{ ways}}$ (2)

Note that expressions (1) and (2) are identical. But a regular tetrahedron is a three dimensional body that can be rotated along its axes of symmetry and the spectator can visualize different arrangements of the colors. For instance, selecting a group of four colors (A, B, C, D), a trihedral containing the colors A, B and C can be colored in two different orders:

A B C or C B A.

Thus, painting with four colors, every group produces two different possibilities (two different arrangements of the same four colors that produce different presentations for the observer), and, consequently, there are 35 x 2 = 70 distinct ways to paint the tetrahedron. Then, the final result will be

$7 + 63 + 105 + 70 = \underline{245 \text{ different ways}}.$

Section III. Three Dimensions

Ordinary Substitutions. Ordinary substitutions are operations that permit one to interrelate the n distinct elements of a given ordinary permutation with the same n distinct elements of another ordinary permutation. Therefore [illegible] substitutions essentially deal with permutations [illegible] any static observer to track [illegible] interposition of the [illegible] body when it is [illegible] its axes of symmetry. [illegible] the n distinct [illegible], the substitution operation [illegible] initial basic permutation [illegible] arrangements that can [illegible]

[illegible] permutation, from which the [illegible] called numerator. Every [illegible] the n! possible [illegible] numerator, because in the [illegible] represented, the [illegible] the numerator in such a way [illegible] between pairs of elements can [illegible] instance, given 6 distinct [illegible] an initial arrangement could be [illegible] This first selected permutation [illegible] the basic point of reference: the

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numerator. Those 6 elements can be arranged in $P_6 = 6!$ $= 720$ different ways (denumerators), every one of which can be interrelated with the numerator. When a particular relationship, such as $\begin{pmatrix} a & b & c & d & e & f \\ a & f & d & e & c & b \end{pmatrix}$ is established

the meaning is that

the initial element <u>a</u> should be now substituted by itself
the initial element <u>b</u> should be now substituted by <u>f</u>
the initial element <u>c</u> should be now substituted by <u>d</u>
the initial element <u>d</u> should be now substituted by <u>e</u>
the initial element <u>e</u> should be now substituted by <u>c</u>
the initial element <u>f</u> should be now substituted by <u>b</u>

As one can see, element <u>a</u> is related to itself and, therefore, it is going to be substituted by itself. Moreover, the possibility exists that every single element is related to itself. In this case, the substitution performed is <u>not</u> going to provide a different arrangement. The name for this type of substitution is the identity substitution.

<u>Example</u>. Given three distinct elements a, b, c, a numerator could be a, b, c, and all possible 6 substitutions ($P_3 = 6$) are:

$\begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix} \longrightarrow$ Identity Substitution

$\begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}$

$\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}$

$\begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$

$\begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}$

$\begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}$

Product of Substitutions. It is important to define now what is understood by the concept product of substitutions. What this expression really means is the result of applying two consecutive substitutions to the same initial arrangement. For instance, given the initial permutation M = 1234, if two substitutions A and B, such that

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

are consecutively applied to M, the product will be

$$P = A \cdot B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix},$$

where

element 1 was substituted by 4 in A, and 4 by 3 in B

element 2 was substituted by 1 in A, and 1 by 2 in B

element 3 was substituted by 2 in A, and 2 by 1 in B

element 4 was substituted by 3 in A, and 3 by 4 in B.

What has been done here is a double substitution. But be careful, because the product of substitutions do <u>not</u> obey the commutative law when the two factors A and B have elements in common. Thus, because factors

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

have elements in common, is not the same to multiply A x B than B x A,

$$A \times B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

and

$$B \times A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

<u>Example</u>. Given $S = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix}$ multiply S x T and T x S

<u>Solution</u>.

$$S \times T = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix}$$

$$T \times S = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

Example. Given

$$S = \begin{pmatrix} a & b & c & d & e \\ b & c & e & a & d \end{pmatrix} \text{ and } T = \begin{pmatrix} a & b & c & d & e \\ c & b & d & e & a \end{pmatrix}$$

prove that $S \times T \neq T \times S$.

Solution. Both factors S and T have elements in common.

$$S \times T = \begin{pmatrix} a & b & c & d & e \\ b & c & e & a & d \end{pmatrix} \times \begin{pmatrix} a & b & c & d & e \\ c & b & d & e & a \end{pmatrix}$$

$$= \begin{pmatrix} a & b & c & d & e \\ b & d & a & c & e \end{pmatrix}, \text{ but}$$

$$T \times S = \begin{pmatrix} a & b & c & d & e \\ c & b & d & e & a \end{pmatrix} \times \begin{pmatrix} a & b & c & d & e \\ b & c & e & a & d \end{pmatrix}$$

$$= \begin{pmatrix} a & b & c & d & e \\ e & c & a & d & b \end{pmatrix}$$

Q.E.D.

Commutable Substitutions. When the product of two substitutions enjoys the commutative property, that is to say, when

$$S \times T = T \times S,$$

these substitutions are called commutable. That happens when both substitutions do not have elements in common. For example, given

$$S = \begin{pmatrix} 1 & 2 & 5 \\ 5 & 1 & 2 \end{pmatrix} \text{ and } T = \begin{pmatrix} 3 & 9 & 8 & 7 \\ 7 & 8 & 9 & 3 \end{pmatrix}$$

$$S \times T = \begin{pmatrix} 1 & 2 & 5 \\ 5 & 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 9 & 8 & 7 \\ 7 & 8 & 9 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 & 3 & 9 & 8 & 7 \\ 5 & 1 & 2 & 7 & 8 & 9 & 3 \end{pmatrix};$$

$$T \times S = \begin{pmatrix} 3 & 9 & 8 & 7 \\ 7 & 8 & 9 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 5 \\ 5 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 9 & 8 & 7 & 1 & 2 & 5 \\ 7 & 8 & 9 & 3 & 5 & 1 & 2 \end{pmatrix};$$

$$S \times T = T \times S = \begin{pmatrix} 1 & 2 & 5 & 3 & 9 & 8 & 7 \\ 5 & 1 & 2 & 7 & 8 & 9 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 9 & 8 & 7 & 1 & 2 & 5 \\ 7 & 8 & 9 & 3 & 5 & 1 & 2 \end{pmatrix}.$$

Effectively, as S does not have any element of T, and T does not have any element of S, neither S can modify T, nor T can modify S. That is to say, for multiplying two or more substitutions that do not have common elements, it is enough to yuxtapose in any other the pairs of elements which compose all of them. Therefore, the order of the factors does not change the product.

So far, only ordinary permutations have been mentioned when the notion of substitutions have been introduced. Recall that ordinary permutations only deal with different arrangements of the same elements in a row, which implies there are first and last elements. This is a one dimensional approach that is not valid when the problem is presented in two and three dimensions. Thus, a bridge has to be constructed that allows one to pass from one dimension (ordinary permutations and ordinary substitutions) to two dimensional problems (circular permutations and

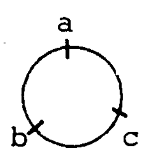
circular substitutions) and from two dimensions to three dimensional problems (cycle index and pattern inventory).

To develop a feeling for such notions, consider the following problems:

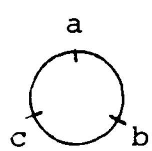
1. <u>One dimensional problem</u>. In how many different ways can three soldiers a, b, and c be arranged in a column? The solution is obvious, $P_3 = 3! = 6$ ways.

2. <u>Two dimensional problem</u>. In how many different ways can three persons a, b, and c sit around a circular table? The solution is easy too, $PC_3 = P_{3-1} = P_2 = 2! = 2$.

3. <u>Three dimensional problem</u>. In how many different ways can three distinct colored beads be strung on a string to form a necklace? The solution given by $PC_3 = P_2 = 2$ has to be divided by 2 because the necklace can be rotated along its diameter, and permutations



and



become identical.

But imagine how difficult it is to visualize three dimensional problems when, for instance, instead of having three distinct colored beads, there are $n > 3$ beads and, in addition, some of them are repeated. A procedure for solving these cases will be explained in the remaining pages of this chapter.

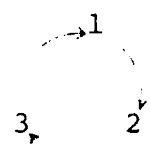
Circular Substitutions or Cycles. Circular substitutions or cycles are a particular case of ordinary substitutions, in the same manner that circular permutations can be considered a particular case of ordinary permutations.

Given a set of $\underline{n}$ distinct elements a_1, a_2, a_3, ..., a_n, a cycle or circular substitution takes place if every element in the denumerator is exactly the same as the element to its right in the numerator, and the last one in the denumerator is the first one in the numerator. Thus,

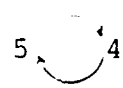
$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_{n-1} & a_n \\ a_2 & a_3 & a_4 & a_5 & \cdots & a_n & a_1 \end{pmatrix}$$

The following expressions represent circular substitutions or cycles:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\ 2\ 3)$$



$$\begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix} = (4\ 5)$$



$$\begin{pmatrix} 6 \\ 6 \end{pmatrix} = (6)$$

Identity substitution

A circular substitution or cycle of length K (number of elements) can be written in K different ways by cyclically permuting the elements composing the cycle.

Here they are, for instance, the 9 equivalent ways to represent a cycle of 9 elements:

1st $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 8 & 7 & 9 & 2 & 4 & 1 & 6 & 3 \end{pmatrix} = (1\ 5\ 2\ 8\ 6\ 4\ 9\ 3\ 7)$

2nd $\begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 \\ 8 & 7 & 9 & 2 & 4 & 1 & 6 & 3 & 5 \end{pmatrix} = (2\ 8\ 6\ 4\ 9\ 3\ 7\ 1\ 5)$

3rd $\begin{pmatrix} 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 & 2 \\ 7 & 9 & 2 & 4 & 1 & 6 & 3 & 5 & 8 \end{pmatrix} = (3\ 7\ 1\ 5\ 2\ 8\ 6\ 4\ 9)$

4th $\begin{pmatrix} 4 & 5 & 6 & 7 & 8 & 9 & 1 & 2 & 3 \\ 9 & 2 & 4 & 1 & 6 & 3 & 5 & 8 & 7 \end{pmatrix} = (4\ 9\ 3\ 7\ 1\ 5\ 2\ 8\ 6)$

5th $\begin{pmatrix} 5 & 6 & 7 & 8 & 9 & 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 6 & 3 & 5 & 8 & 7 & 9 \end{pmatrix} = (5\ 2\ 8\ 6\ 4\ 9\ 3\ 7\ 1)$

6th $\begin{pmatrix} 6 & 7 & 8 & 9 & 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 6 & 3 & 5 & 8 & 7 & 9 & 2 \end{pmatrix} = (6\ 4\ 9\ 3\ 7\ 1\ 5\ 2\ 8)$

7th $\begin{pmatrix} 7 & 8 & 9 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 3 & 5 & 8 & 7 & 9 & 2 & 4 \end{pmatrix} = (7\ 1\ 5\ 2\ 8\ 6\ 4\ 9\ 3)$

8th $\begin{pmatrix} 8 & 9 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 3 & 5 & 8 & 7 & 9 & 2 & 4 & 1 \end{pmatrix} = (8\ 6\ 4\ 9\ 3\ 7\ 1\ 5\ 2)$

9th $\begin{pmatrix} 9 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 8 & 7 & 9 & 2 & 4 & 1 & 6 \end{pmatrix} = (9\ 3\ 7\ 1\ 5\ 2\ 8\ 6\ 4)$

Breaking Down Ordinary Substitutions into Cycles.

Any ordinary substitution (no circular) can be broken down into a unique product of circular substitutions without common elements. For example,

$$\begin{pmatrix}1&2&3&4&5&6&7&8&9\\4&8&2&6&1&7&5&9&3\end{pmatrix} = \begin{pmatrix}1&4&6&7&5\\4&6&7&5&1\end{pmatrix} \times \begin{pmatrix}2&8&9&3\\8&9&3&2\end{pmatrix}$$

The multiplication cannot be done because any element in any of these two cycles is repeated.

$$\begin{pmatrix}1&2&3&4&5&6&7\\2&4&5&1&7&6&3\end{pmatrix} = \begin{pmatrix}1&2&4\\2&4&1\end{pmatrix} \times \begin{pmatrix}3&5&7\\5&7&3\end{pmatrix} \times \begin{pmatrix}6\\6\end{pmatrix} = \begin{pmatrix}1&2&4\\2&4&1\end{pmatrix} \times \begin{pmatrix}3&5&7\\5&7&3\end{pmatrix}$$

The reasoning is the same as before. Notice here that you have the option of not showing the identity substitution $\begin{pmatrix}6\\6\end{pmatrix}$ which means that the substitution of 6 by itself is implied.

$$\begin{pmatrix}5&1&2&7&8&9&3\\1&2&5&3&9&8&7\end{pmatrix} = \begin{pmatrix}5&1&2\\1&2&5\end{pmatrix} \times \begin{pmatrix}7&3\\3&7\end{pmatrix} \times \begin{pmatrix}8&9\\9&8\end{pmatrix}$$

$$= \begin{pmatrix}7&3\\3&7\end{pmatrix} \times \begin{pmatrix}5&1&2\\1&2&5\end{pmatrix} \times \begin{pmatrix}8&9\\9&8\end{pmatrix}$$

$$= \begin{pmatrix}8&9\\9&8\end{pmatrix} \times \begin{pmatrix}7&3\\3&7\end{pmatrix} \times \begin{pmatrix}5&1&2\\1&2&5\end{pmatrix}$$

The reasoning is the same as before. Notice here that the product has the commutative property (the order of the factors does not change the result) because the cycles do not have elements in common.

In general, given any ordinary substitution (no circular), it can be represented in product form of factors, being such factors the circular substitutions into which the given ordinary substitution can be decomposed. That is to say,

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_\alpha & b_1 & b_2 & b_3 & \cdots & b_\beta & \cdots\cdots & \ell_1 & \ell_2 & \ell_3 & \cdots & \ell_\lambda \\ a_2 & a_3 & a_4 & \cdots & a_1 & b_2 & b_3 & b_4 & \cdots & b_1 & \cdots\cdots & \ell_2 & \ell_3 & \ell_4 & \cdots & \ell_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_\alpha \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix} \times \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_\beta \\ b_2 & b_3 & b_4 & \cdots & b_1 \end{pmatrix} \times \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 & \cdots & \ell_\lambda \\ \ell_2 & \ell_3 & \ell_4 & \cdots & \ell_1 \end{pmatrix}$$

$$= (a_1\ a_2\ a_3\ \cdots\ a_\alpha) \times (b_1\ b_2\ b_3\ \cdots\ b_\beta) \times \cdots \times (\ell_1 \ell_2 \ell_3 \cdots \ell_\lambda)$$

Notice the abbreviated way used in the last expression to symbolize the circular substitutions or cycles.

It is important to mention at this point a different type of notation for representing the breaking down of ordinary substitutions into cycles. For instance,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 5 & 1 & 7 & 6 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 5 & 7 \\ 5 & 7 & 3 \end{pmatrix} \times \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

$$= (6) \times (1\ 2\ 4) \times (3\ 5\ 7)$$

$$= f_1^1 \cdot f_3^1 \cdot f_3^1 = f_1^1 \cdot f_3^2$$

What does $f_1^1 \cdot f_2^2$ mean? It is just a brief and condensed way

to say that the ordinary substitution

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 5 & 1 & 7 & 6 & 3 \end{pmatrix}$ has been broken down into one cycle of

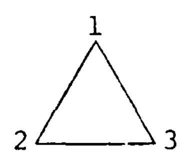
length 1, $(6) = f^{1=\#\text{ of cycles}}_{1\,=\,\text{length of every cycle.}}$

plus two cycles of length 3, $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{pmatrix}\begin{pmatrix} 3 & 5 & 7 \\ 5 & 7 & 3 \end{pmatrix}$

$$= f_3^1 \cdot f_3^1 = f^{2\,=\,\#\text{ of cycles}}_{3\,=\,\text{length of every cycle.}}$$

In general, the expression f_q^p represents that there are p cycles with q elements each. But notice that this notation does not tell you anything about the specific elements that are interrelated. You lack information here. The only thing that you know is the general cyclical structure of an ordinary substitution.

The Cycle Index. Imagine an equilateral triangle that has its three vertices numbered with one different number each. Suppose you are a statical observer who stands just in front of one of the triangle faces. You are seeing a particular arrangement of the three numbers at this starting moment, such as

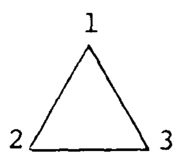


At this point, the triangle is static. But later on, the triangle is going to start rotating around itself, symmetrically, around its axes or points of symmetry.

Every time the triangle completes any of those symmetrical movements, you would like to compute how many different configurations, or distinct arrangements of the numbers, you have seen. This is done in order to be able to tell at the end, when all possible rotations have been performed, the total amount of distinct arrangements of the vertices that have appeared during the whole process.

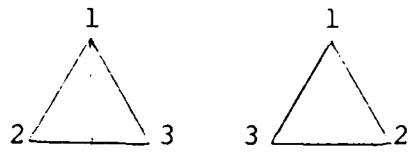
The theory of substitutions will play a fundamental role here for tracking in the space the position of the three vertices (three numbers) with respect to its initial positions.

The initial position of the vertices is given when the triangle has not yet started any movement. Thus, no rotation, initial position and identity substitution go together.

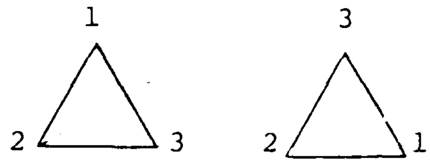


$$\binom{1\ 2\ 3}{1\ 2\ 3} = \binom{1}{1}\binom{2}{2}\binom{3}{3} = (1)\ (2)\ (3) = f_1^3$$

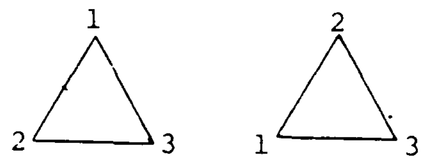
Later on the equilateral triangle is flipped 180° around the three axes going through vertices 1, 2, 3 and through the center of the respective opposite edges. Thus,



$$\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix} \begin{pmatrix}2 & 3\\ 3 & 2\end{pmatrix} = (1)\ (2\ 3) = f_1^1\ f_2^1$$



$$\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix} = \begin{pmatrix}2\\ 2\end{pmatrix} \begin{pmatrix}1 & 3\\ 3 & 1\end{pmatrix} = (2)\ (13) = f_1^1\ f_2^1$$



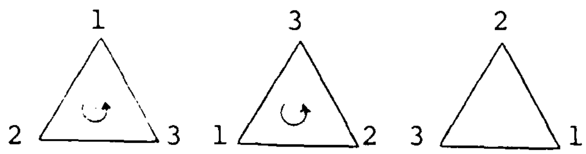
$$\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix} = \begin{pmatrix}3\\ 3\end{pmatrix} \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix} = (3)\ (1\ 2) = f_1^1\ f_2^1$$

When the three rotations are over, it can be said that the three have the same structure $f_1^1\ f_2^1$ in the sense that the three ordinary substitutions can be expressed by the product of one cycle of length 1 (f_1^1) by one cycle of length 2 (f_2^1). As the same thing happens three times, the general structure for the three 180° rotations together will be $3f_1^1\ f_2^1$.

Notice again that with this notation you are losing information, because such notation, by itself,

does not show the specific vertices that are cyclically interrelated. A solution for clarifying such temporal lack of clearness will be given by the pattern inventory.

The equilateral triangle can also be rotated around its center of gravity, $120° = \frac{360°}{3}$ three times



This is what the spectator can see after every rotation has finished:

after first 120° rotation, $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1\ 3\ 2) = f_3^1$; and

after second 120° rotation = 240° total rotation,

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\ 2\ 3) = f_3^1$.

As it has been analyzed, the static spectator has rotated the equilateral triangle around all its axes (or points) of symmetry. The 6 total number of rotations performed have produced the following configurations:

1. (1)(2)(3) $= f_1^3$

2. (1)(2 3) $= f_1^1 f_2^1$

(2)(1 3) $= f_1^1 f_2^1$ $= 3f_1^1 f_2^1$

(3)(1 2) $= f_1^1 f_2^1$

3. (1 3 2) $= f_3^1$

$= \underline{2f_3^1}$

(1 2 3) $= f_3^1$

Total number of rotations performed = 6

By definition, the cycle index is a division which the denumerator is the total number of rotations performed, and the numerator is the sum of all the arrangements that have appeared after all rotations have finished.

The cycle index for the equilateral triangle will be:

$$\frac{f_1^3 + 3f_1^1 f_2 + 2f_3^1}{6}$$

The cycle index is a very abstract formula that compiles cycles, but, as a chrysalis, it explodes plenty of meaningfulness when the pattern inventory notion is applied to it. So, what will be the value of the cycle index? The cycle index will be the base for counting how many different

configurations, with respect to the initial one, the static observer sees when the body has been rotated around all its axes of symmetry.

The cycle index is, in a certain sense, the unit because it is a fraction whose numerator and denumerator represent the same quantities. The denumerator is just the number of total rotations performed, and the numerator provides a picture of the different types of substitutions found during the rotations. That is why

$$\begin{array}{l}\text{Cycle index}\\ \text{for equilateral}\\ \text{triangle}\end{array} = \frac{6 \text{ ordinary substitutions}}{6 \text{ rotations performed}}$$

$$= \frac{1 \text{ substitution } f_1^3 + 3 \text{ substitutions } f_1^1 f_2^1 + 2 \text{ substitutions } f_3^1}{6 \text{ rotations performed}}$$

The Pattern Inventory. Any ordinary substitution can be cyclically represented by mean of a product of circular substitutions. As far as the equilateral triangle is concerned, it has been already proved that:

Rotations	Substitutions		Cycles of the Vertices		Condensed Form	
0°	$\binom{1\ 2\ 3}{1\ 2\ 3}$	=	(1) (2) (3)	=		f_1^3
180°	$\binom{1\ 2\ 3}{1\ 3\ 2}$	=	(1) (2 3)	=	$f_1^1 f_2^1$	
180°	$\binom{1\ 2\ 3}{3\ 2\ 1}$	=	(2) (1 3)	=	$f_1^1 f_2^1$	$3f_1^1 f_2^1$
180°	$\binom{1\ 2\ 3}{2\ 1\ 3}$	=	(3) (1 2)	=	$f_1^1 f_2^1$	
120°	$\binom{1\ 2\ 3}{3\ 1\ 2}$	=	(1 3 2)	=	f_3^1	$2f_3^1$
240°	$\binom{1\ 2\ 3}{2\ 3\ 1}$	=	(1 2 3)	=	f_3^1	

Cycle Index for Equilateral Triangle $= \dfrac{f_1^3 + 3f_1^1 f_2^1 + 2f_3^1}{6}$

One time that the cycle index has been calculated, each of those abstract terms $f_{q=\text{length of each cycle}}^{p=\#\text{ of cycles}}$ is replaced by the called pattern inventory. Thus,

$$f_q^p \longrightarrow (x^q + y^q + z^q + \ldots)^p,$$

being x, y, z, ... etc., the different variables playing a role in the specific problem to be solved. For instance, if the problem consists of calculating in how many different ways can 6 beads (1 red, 1 green, 4 yellow) be attached to the vertices of a regular hexagon, the variables are three (red, green, yellow).

Application Problems.

Problem No. 1. In how many different ways can the vertices of an equilateral triangle be painted if three different colors (a, b, c) are available, and repeated selection of the colors is allowed?

Solution. The cycle index is $\frac{1}{6}(f_1^3 + 3f_1^1 f_2^1 + 2f_3^1)$.

There are three variables (a, b, c), so the replacements to be performed in the cycle index, in order to obtain the expanded cycle index are:

$$f_1^3 = (a + b + c)^3$$

$$3 \cdot f_1^1 \cdot f_2^1 = 3(a + b + c) \cdot (a^2 + b^2 + c^2)$$

$$2 \cdot f_3^1 = 2(a^3 + b^3 + c^3)$$

That means,

$$\text{Expanded Cycle Index for the Equilateral Triangle} = \frac{(a+b+c)^3+3(a+b+c)(a^2+b^2+c^2)+2(a^3+b^3+c^3)}{6}$$

The pattern inventory terms can easily be solved:

$$\text{Term } (a+b+c)^3 = \Sigma \frac{3!}{\alpha!\beta!\gamma!} a^\alpha b^\beta c^\gamma \quad \text{(Leibniz' Formula)}$$

$$\alpha+\beta+\gamma = m = 3;\ n = 3$$

Values for α, β, γ

$$m = 3 = 3 + 0 + 0 \longrightarrow a^3,\ b^3,\ c^3$$

$$m = 3 = 2 + 1 + 0 \longrightarrow a^2b,\ a^2c,\ b^2a,\ b^2c,\ c^2a,\ c^2b$$

$$m = 3 = 1 + 1 + 1 \longrightarrow abc$$

$$(a+b+c)^3 = \frac{3!}{3!0!0!}(a^3+b^3+c^3) + \frac{3!}{2!1!0!}(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b)$$

$$+ \frac{3!}{1!1!1!} \cdot (a\,bc)$$

$$= a^3+b^3+c^3+3(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b) + 6\,abc.$$

$$\text{Term 3 x } (a+b+c)(a^2+b^2+c^2) = 3(a^3+b^3+c^3)$$

$$+ 3(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b).$$

$$\text{Term 2 x } (a^3+b^3+c^3) = 2(a^3+b^3+c^3).$$

Adding the three terms and dividing by 6,

$$\begin{array}{l}\text{Expanded}\\ \text{Cycle}\\ \text{Index}\end{array} = \frac{6(a^3+b^3+c^3) + 6(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b) + abc}{6}$$

$$= a^3+b^3+c^3+a^2b+a^2c+b^2a+b^2c+c^2a+c^2b+abc.$$

There are 10 terms here. Thus, there are 10 different ways of painting the vertices of the equilateral triangle with repeated selection of colors:

	Vertex 1	Vertex 2	Vertex 3
1st	a	a	a
2nd	b	b	b
3rd	c	c	c
4th	a	a	b
5th	a	a	c
6th	b	b	a
7th	b	b	c
8th	c	c	a
9th	c	c	b
10th	a	b	c

A consideration for computing the number of different ways of painting the vertices of the equilateral triangle using the

$$\text{Expanded Cycle Index} = a^3+b^3+c^3+a^2b+a^2c+b^2a+b^2c+c^2a+c^2b+abc,$$

is the following:

As a, b, and c can be equally pondered, because the three different colors a, b, and c are used the same amount of times, it is convenient to make $a = b = c = 1$ in order to obtain a value for the expanded cycle index. Thus, this value is 10, which indicates that there are 10 different ways to paint the equilateral triangle, using three different colors, when repeated selection is allowed.

The question that immediately occurs is why does the replacement of the pattern inventory into the cycle index produce such a result. The explanation will be given in the following paragraphs. Such explanation will be provided along with the resolution of several problems. As the explanation of those typical problems is given, everyone should be able to understand why the pattern inventory is a convenient albegraic expression that usually drives one to have to multiply two or more polynomials, all containing the same variables. The result of that convenient multiplication produces all possible combinations of those variables, the variables of the particular problem to be solved.

Why has f_1^3 been replaced by $(a+b+c)^3$? Recall that f_1^3 means that there are three independent vertices, (1) (2) (3); each one is a cycle by itself:

$$1 \circlearrowright \qquad 2 \circlearrowright \qquad 3 \circlearrowright$$

These three vertices (1, 2, 3) have to be painted with three different colors (a, b, c) that can be repetitively selected. Representing the vertices by three independent boxes $\square\ \square\ \square$, they can be painted

$$a^3,\ b^3,\ c^3;\ a^2b,\ a^2c,\ b^2a,\ b^2c,\ c^2a,\ c^2b;\ abc.$$

a^3 indicates that the three boxes are painted with color _a_.

a^2b indicates that two boxes are painted with color a, and the other box with color b.

abc indicates that every box is painted with a different color.

Table 1 shows the assignments of colors (a, b, c) to vertices (1, 2, 3). Referring to Table 1, one can see the result is $a^3+b^3+c^3+3\ (a^2b+a^2c+b^2a+b^2c+c^2a+c^2b) + 6abc$, exactly the same result as was gotten by the development of the term $(a+b+c)^3$ using the Leibniz formula.

Analytically it has been proved why f_1^3 was replaced by $(a+b+c)^3$. Now, the question is similar: why $3f_1^1f_2^1$ was replaced by $3(a+b+c)(a^2+b^2+c^2)$?

Recall that $3f_1^1f_2^1$ means here that when three 180° rotations were completed around three homologous axes of symmetry, the spectator saw three different arrangements of the three vertices, each of them broken into two disjoint cycles; that means, into two cycles with no common elements.

Graphically,

$$3f_1^1f_2^1 \begin{cases} (1)\ (2\ 3) \\ (2)\ (1\ 3) \\ (3)\ (1\ 2) \end{cases}$$

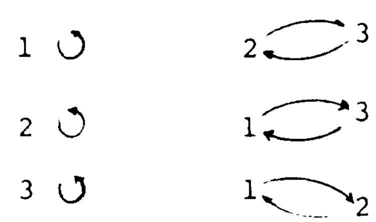


TABLE 1

ASSIGNMENTS OF COLORS (a, b, c) TO VERTICES (1, 2, 3)

	1	2	3	
a^3	a	a	a	a^3
b^3	b	b	b	b^3
c^3	c	c	c	c^3
a^2b	a	a	b	
a^2b	a	b	a	$3a^2b$
a^2b	b	a	a	
a^2c	a	a	c	
a^2c	a	c	a	$3a^2c$
a^2c	c	a	a	
b^2a	b	b	a	
b^2a	b	a	b	$3b^2a$
b^2a	a	b	b	
b^2c	b	b	c	
b^2c	b	c	b	$3b^2c$
b^2c	c	b	b	
c^2a	c	c	a	
c^2a	c	a	c	$3c^2a$
c^2a	a	c	c	
c^2b	c	c	b	
c^2b	c	b	c	$3c^2b$
c^2b	b	c	c	

TABLE 1--*Continued*

	1	2	3	
	□	□	□	
abc	a	b	c	
abc	a	c	b	
abc	a	b	c	6abc
abc	c	b	a	
abc	a	c	b	
abc	b	c	a	

Representing the vertices by three boxes, one independent box $\square$, two dependent and joined boxes $\square\square$, they can be painted in this way:

a^3, b^3, c^3;

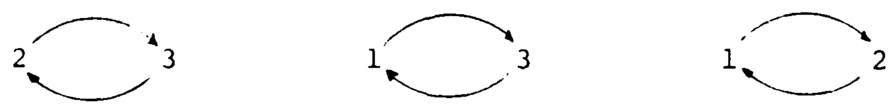
ab^2, ac^2; ba^2, bc^2; ca^2, cb^2;

where

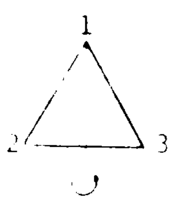
a^3 means that the three boxes are painted with the same color a.

ab^2 means that one box is painted with a and the two joined boxes with the same color b.

Notice that the joined boxes $\square\square$ are always painted both with the same color (aa, bb, cc). That happens because the circular substitutions or cycles



imply that the first color that is given to any vertex included into a cycle is cyclically transmitted to the other interrelated vertices. Thus, the color assigned in the first arrangement to vertex 2, when the triangle is flipped around its height over the side 2 3, [illegible]



The color assigned to vertex 2
is reassigned to vertex 3,
and vice versa.

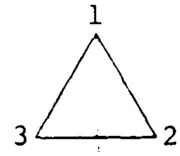


Table 2 shows all possible assignments of colors (a, b, c) to the vertices (1, 2, 3). The table result is:

$$3(a^2+b^2+c^2) + 3(ab^2+ac^2+ba^2+bc^2+ca^2+cb^2),$$

exactly the same result as was gotten by the development of the term 3 x $(a+b+c)(a^2+b^2+c^2)$.

Analytically it has been proven why $3f_1^1 f_2^1$ was replaced by $3(a+b+c)(a^2+b^2+c^2)$. Similarly, the question now is why $2f_3^1$ was replaced by $2(a^3+b^3+c^3)$.

Recall that $2f_3^1$ means here that when two consecutive 120° rotations were completed around the center of gravity of the triangle, the spectator saw two different arrangements of the three vertices, each of them is cyclical. Graphically,

$$2f_3^1 \left\{ \begin{array}{l} (1\ 3\ 2) \\ (1\ 2\ 3) \end{array} \right.$$

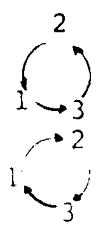


TABLE 2

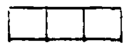
ASSIGNMENTS OF COLORS (a, b, c) TO VERTICES (1, 2, 3)

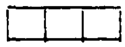
1	2 3	
a	aa	a^3
b	bb	b^3
c	cc	c^3
a	bb	ab^2
a	cc	ac^2
b	aa	ba^2
b	cc	bc^2
c	aa	ca^2
c	bb	cb^2

2	1 3	
a	aa	a^3
b	bb	b^3
c	cc	c^3
a	bb	ab^2
a	cc	ac^2
b	aa	ba^2
b	cc	bc^2
c	aa	ca^2
c	bb	cb^2

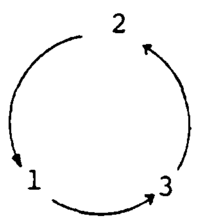
TABLE 2--Continued

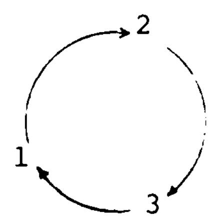
3	1 2	
a	aa	a^3
b	bb	b^3
c	cc	c^3
a	bb	ab^2
a	cc	ac^2
b	aa	ba^2
b	cc	bc^2
c	aa	ca^2
c	bb	cb^2

Representing the vertices by three joined boxes  and , these boxes can only be painted

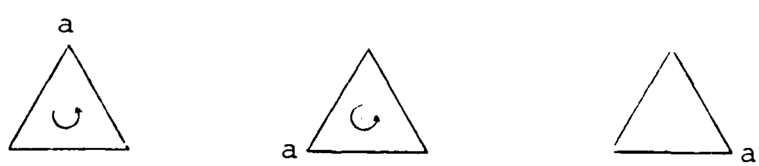
		
aaa	aaa	$2a^3$
bbb	bbb	$2b^3$
ccc	ccc	$2c^3$

The reason why only one single color can be seen every time the triangle is rotated is because of the cyclical structure of the vertices.





For instance, if color a was given to vertex 1 in the first assignment, when the triangle is rotated 120° around its center of gravity, the same color a goes to vertex 2.



And when a new rotation of 120° is performed, the same color a goes to position 3. Therefore, it has been justified why $2f_3^1$ was replaced by $2(a^3+b^3+c^3)$.

On the other hand, it is interesting to remember that the cycle index is really the unit,

$$\begin{array}{l}\text{Cycle Index}\\ \text{for Equilateral}\\ \text{Triangle}\end{array} = \frac{\text{6 ordinary substitutions}}{\text{6 rotations performed}} = \frac{1\cdot f_1^3+3f_1^1f_2^1+2f_3^1}{6}$$

$$1 + 3 + 2 = 6$$

At this point it should be clear in a general way, why the terms of the condensed cycle index expression is properly replaced by the pattern inventory

$$f_q^p \text{ really means } (a^q+b^q+\ldots+z^q)^p$$

because developing the p power of the polynomial $(a^q+b^q+\ldots+z^q)^p$ by the Leibniz Formula

$$\Sigma \quad \frac{p!}{\alpha!\beta!\ldots\lambda!}(a^q)^\alpha(b^q)^\beta\ldots(z^q)^\lambda$$

$$\alpha+\beta+\ldots+\lambda=p$$

the different factors (a^q, b^q, ... z^q) involved in every term of such development always have the required length q of the p cycles. Additionally, there are always p factors or cycles in every term, because $\alpha+\beta+\ldots+\lambda=p$.

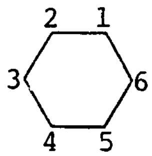
Problem No. 2.

1. Calculate the cycle index of a regular hexagon.

2. In how many different ways can 6 beads (1 red, 1 green and 4 yellow) be attached to the vertices of the regular hexagon?

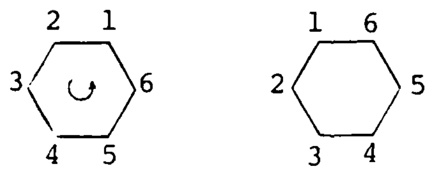
Solution.

1. Cycle index. No rotation. Initial position. Identity substitution.



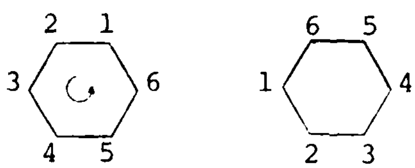
$$\begin{pmatrix}1&2&3&4&5&6\\1&2&3&4&5&6\end{pmatrix} = (1)(2)(3)(4)(5)(6) = f_1^6 \qquad \begin{array}{l} 6 = \text{\# of cycles} \\ 1 = \text{length of each cycle} \end{array}$$

Rotating $60° = \frac{360°}{6}$ around its center of symmetry,



$$\begin{pmatrix}1&2&3&4&5&6\\6&1&2&3&4&5\end{pmatrix} = (1\ 2\ 3\ 4\ 5\ 6) = f_6^1$$

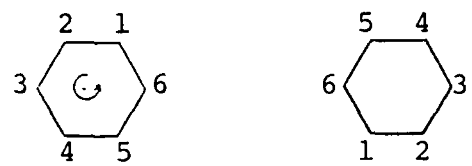
Rotating 120° around its center of symmetry,



$$\begin{pmatrix}1&2&3&4&5&6\\5&6&1&2&3&4\end{pmatrix} = \begin{pmatrix}1&5&3\\5&3&1\end{pmatrix}\begin{pmatrix}2&6&4\\6&4&2\end{pmatrix} = (1\ 5\ 3)\ (2\ 6\ 4)$$

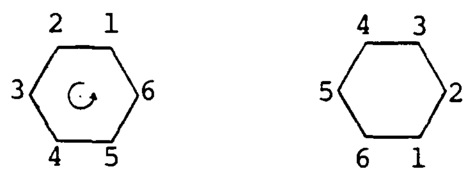
$$= f_3^1 \cdot f_3^1 = f_3^2$$

Rotating 180° around its center of symmetry,



$$\begin{pmatrix}1&2&3&4&5&6\\4&5&6&1&2&3\end{pmatrix} = \begin{pmatrix}1&4\\4&1\end{pmatrix}\ \begin{pmatrix}2&5\\5&2\end{pmatrix}\ \begin{pmatrix}3&6\\6&3\end{pmatrix} = f_2^3$$

Rotating 240° around its center of symmetry,

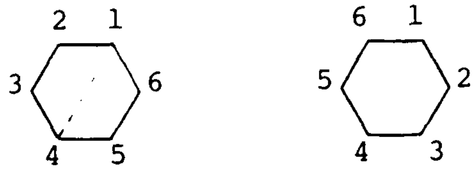


$$\begin{pmatrix}1&2&3&4&5&6\\3&4&5&6&1&2\end{pmatrix} = \begin{pmatrix}1&3&5\\3&5&1\end{pmatrix}\ \begin{pmatrix}2&4&6\\4&6&2\end{pmatrix} = f_3^1 \cdot f_3^1 = f_3^2$$

Rotating 300° around its center of symmetry,

$$\begin{pmatrix}1&2&3&4&5&6\\2&3&4&5&6&1\end{pmatrix} = f_6^1$$

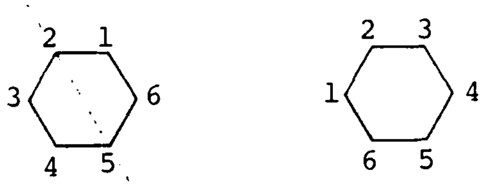
Rotating 180° around the axe that goes through vertices 1 and 4,



$$\begin{pmatrix}1&2&3&4&5&6\\1&6&5&4&3&2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}4\\4\end{pmatrix}\begin{pmatrix}2&6\\6&2\end{pmatrix}\begin{pmatrix}3&5\\5&3\end{pmatrix}$$

$$= (1)\ (4)\ (26)(35) = f_1^1 f_1^1 f_2^1 f_2^1 = f_1^2 f_2^2$$

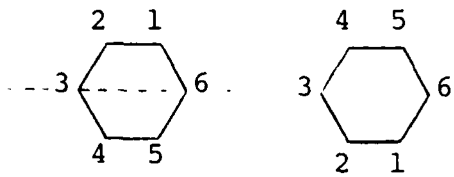
Rotating 180° around the axe that goes through vertices 2 and 5,



$$\begin{pmatrix}1&2&3&4&5&6\\3&2&1&6&5&4\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}\begin{pmatrix}5\\5\end{pmatrix}\begin{pmatrix}1&3\\3&1\end{pmatrix}\begin{pmatrix}4&6\\6&4\end{pmatrix}$$

$$= (2)\ (5)\ (13)\ (46) = f_1^1 f_1^1 f_2^1 f_2^1 = f_1^2 f_2^2$$

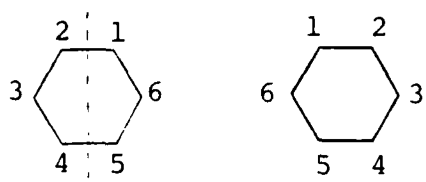
Rotating 180° around the axe that goes through vertices 3 and 6,



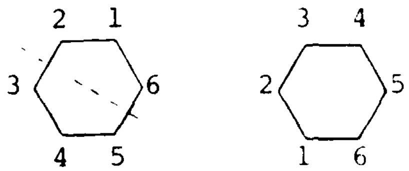
$$\begin{pmatrix}1&2&3&4&5&6\\5&4&3&2&1&6\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix} \begin{pmatrix}6\\6\end{pmatrix} \begin{pmatrix}1&5\\5&1\end{pmatrix} \begin{pmatrix}2&4\\4&2\end{pmatrix}$$

$$= (3)\ (6)\ (1\ 5)\ (2\ 4) = f_1^1 f_1^1 f_2^1 f_2^1 = f_1^2 f_2^2$$

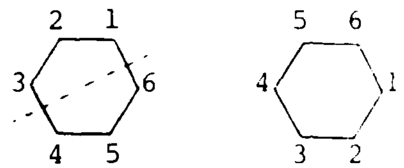
Rotating 180° around the three axes going through the middle points of opposite edges,



$$\begin{pmatrix}1&2&3&4&5&6\\2&1&6&5&4&3\end{pmatrix} = \begin{pmatrix}1&2\\2&1\end{pmatrix} \begin{pmatrix}3&6\\6&3\end{pmatrix} \begin{pmatrix}4&5\\5&4\end{pmatrix} = (1\ 2)\ (3\ 6)\ (4\ 5) = f_2^3$$



$$\begin{pmatrix}1&2&3&4&5&6\\4&3&2&1&6&5\end{pmatrix} = \begin{pmatrix}1&4\\4&1\end{pmatrix}\begin{pmatrix}2&3\\3&2\end{pmatrix}\begin{pmatrix}5&6\\6&5\end{pmatrix} = (14)\ (23)\ (56)$$



$$\begin{pmatrix}1&2&3&4&5&6\\6&5&4&3&2&1\end{pmatrix} = \begin{pmatrix}1&6\\6&1\end{pmatrix}\begin{pmatrix}2&5\\5&2\end{pmatrix}\begin{pmatrix}3&4\\4&3\end{pmatrix} = (16)\ (25)\ (34)$$

Total of rotations performed: 12 rotations (cycle index denominator)

Types of circular substitutions or cycles made with these

12 rotations: $f_1^6 + 2f_6^1 + 2f_3^2 + 4f_2^3 + 3f_1^2 f_2^2$ (cycle index numerator).

Thus, the cycle index is

$$\text{Cycle Index for the Regular Hexagon} = \frac{f_1^6 + 2f_6^1 + 2f_3^2 + 4f_2^3 + 3f_1^2 f_2^2}{12}$$

2. Pattern inventory. There are three variables in this problem:

a. red bead
b. green bead
c. yellow bead

Thus, the pattern inventory function will consist in the replaceme.. of the following convenient expression into the cycle index formula,

$$f_q^p \longrightarrow (a^q + b^q + c^q)^p$$

$$\begin{array}{r}\text{Cycle Index} \\ \text{for the Hexagon}\end{array} = \frac{1}{12} \cdot (f_1^6 + 2f_6^1 + 2f_3^2 + 4f_2^3 + 3f_1^2 f_2^2) —$$

$$\begin{array}{r}\text{Expanded Cycle Index} \\ \text{for the Hexagon}\end{array} = \frac{1}{12} \cdot [(a+b+c)^6 + 2(a^6+b^6+c^6)$$

$$+ 2(a^3+b^3+c^3)^2 + 4(a^2+b^2+c^2)^3$$

$$+ 3(a+b+c)^2(a^2+b^2+c^2)^2]$$

The development of the whole expanded cycle index produces a result valid for attaching beads to the vertices with repeated selection. But in this problem there is no chance for repeated selection. So, the only valid structure for this problem is $a\,b\,c^4$ (1 red bead, 1 green bead, and 4 yellow beads).

The term $f_1^6 = (a+b+c)^6$ produces the structure $a\,b\,c^4$ whose coefficient can be obtained applying the Leibniz Formula,

$$(a+b+c)^6 = \Sigma \frac{6!}{\alpha!\beta!\gamma!} a^\alpha b^\beta c^\gamma$$

$$m = 6 = \alpha + \beta + \gamma \text{ ; } n = 3.$$

For $\alpha = 1$, $\beta = 1$, $\gamma = 4$,

$$\frac{6!}{1!1!4!} a^1 b^1 c^4 = 30\, a\, b\, c^4$$

The terms $2f_6^1 = 2(a^6 + b^6 + c^6)$,

$$2f_3^2 = 2(a^3 + b^3 + c^3)^2, \text{ and}$$

$$4f_2^3 = 4(a^2 + b^2 + c^2)^3$$

do not produce the arrangement $a\,b\,c^4$.

The term $3f_1^2 f_2^2 = 3(a+b+c)^2 \cdot (a^2+b^2+c^2)^2$

$$= 3(a^2+b^2+c^2+2ab+2ac+2bc)$$

$$\times\ (a^4+b^4+c^4+2a^2b^2+2a^2c^2+2b^2c^2).$$

From this expression it can be deduced that

$$3 \cdot 2ab \cdot c^4 = 6\, a\, b\, c^4$$

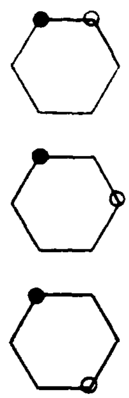
Replacing these two values in the expanded cycle index,

Expanded cycle index for regular hexagon $= \frac{1}{12}(30\,abc^4 + 6\,abc^4 + \ldots\ldots)$

$$= \frac{1}{12}(36\,abc^4 + \ldots\ldots)$$

$$= 3\,abc^4 + \frac{1}{12} \cdot \text{rest of arrangements.}$$

Thus, the solution is 3 different ways. The same solution can be graphically observed



Solution: 3 different arrangements

The reader's concern should be focused again in knowing why the pattern inventory works. Why f_1^6 was replaced by $(a+b+c)^6$?

Recall that f_1^6 signifies that there is a decomposed ordinary substitution into six circular substitutions with no elements in common,

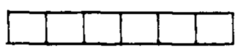
The six beads have to be assigned, one to each box. In how many ways can they be assigned?

$$PR_6^{1,1,4} = \frac{6!}{1!1!4!} = 30$$

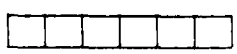
This expression and solution are both identical to those obtained using the Leibniz Formula,

$$\frac{6!}{1!1!4!} a^1 b^1 c^4 = 30\ a\, b\, c^4$$

The logical question to ask next is why f_6^1 was replaced by $(a^6 + b^6 + c^6)$. The reasoning is always the same: what does f_6^1 mean? A circular substitution of length 6.



If f_6^1 is a cycle, the same initial colored bead that was assigned to its first cyclically linked vertex will be repeated through all the six concatenated vertices. That implies that the six boxes should receive the same colored beads (a, b, c):



$$\begin{array}{cccccccc} a & a & a & a & a & a & \longrightarrow & a^6 \\ b & b & b & b & b & b & \longrightarrow & b^6 \\ c & c & c & c & c & d & \longrightarrow & c^6 \end{array}$$

But a constraint in this problem is that there are not six equally colored beads, and repeated selection is not allowed either. Therefore, this computation does not serve any purpose.

Another question that should be asked is what is the rationale for the transformation

$$3f_1^2 f_2^2 \longrightarrow 3(a+b+c)^2 \ (a^2+b^2+c^2)^2$$

Here there are three ordinary substitutions broken down into circular substitutions or cycles.

$$\begin{array}{llllcllll} (1) & (4) & (26) & (35) & \longrightarrow & \square & \square & \square\square & \square\square & \\ (2) & (5) & (13) & (46) & \longrightarrow & \square & \square & \square\square & \square\square & 3f_1^2 f_2^2 \\ (3) & (6) & (15) & (24) & \longrightarrow & \square & \square & \square\square & \square\square & \end{array}$$

Beads a, b, and c can individually go into the single independent boxes. Paired beads aa, bb and cc, identically colored, have to go into the joined boxes because of their cyclical condition. Thus, the product

$$3(a + b + c)^2 \ (a^2 + b^2 + c^2)^2$$

$$= 3(a^2 + b^2 + c^2 + 2ab + 2ac + 2bc)\ (a^4 + b^4 + c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2)$$

$$= 3(a^6+a^2b^4+a^2c^4+2a^4b^2+2a^4c^2+2a^2b^2c^2+\ldots)$$

provides all possible combinations of assignments. For instance,

□	□	□□	□□
a	a	a a	a a
a	a	b b	b b
a	a	c c	c c
a	a	a a	b b
a	a	a a	c c
a	a	b b	c c
b	b	a a	b b
c	c	a a	c c
b	b	a a	a a
c	c	a a	b b
b	b	a a	c c
.....	.....	.....	.....
.....	.....	.....	.....

In the problem, the constraint is

1 bead of red color

1 bead of green color

4 beads of yellow color

and the variables are the three colors

a = color red

b = color green

c = color yellow

Therefore, not every combination of those given by the expanded cycle index can be accepted. Valid replies are only those satisfying the constraint $a^1 b^1 c^4 = a\,b\,c^4$.

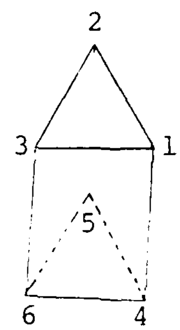
Problem No. 3.

1. Calculate the cycle index of a regular triangular prism.

2. Calculate in how many different ways two distinct colored beads (a and b) can be attached to the vertices of the regular triangular prism.

Solution.

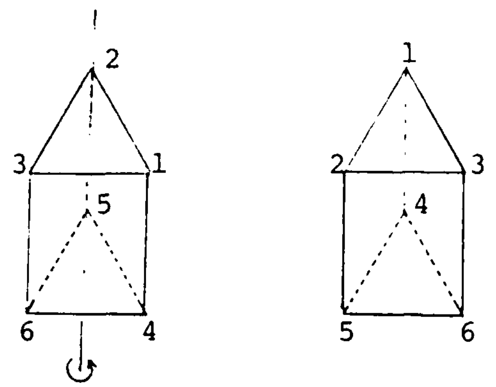
1. Cycle index. No rotation. Initial position. Identity substitution.



$$\binom{1\ 2\ 3\ 4\ 5\ 6}{1\ 2\ 3\ 4\ 5\ 6} = \binom{1}{1}\ \binom{2}{2}\ \binom{3}{3}\ \binom{4}{4}\ \binom{5}{5}\ \binom{6}{6}$$

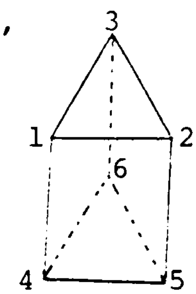
$$= (1)\ (2)\ (3)\ (5)\ (5)\ (6) = f_1^6$$

Rotating $\frac{360°}{3} = 120°$ around a vertical axe that goes through the center of symmetry of both triangular faces,



$$\begin{pmatrix}1&2&3&4&5&6\\3&1&2&6&4&5\end{pmatrix} = \begin{pmatrix}1&3&2\\3&2&1\end{pmatrix}\begin{pmatrix}4&6&5\\6&5&4\end{pmatrix} = (1\ 3\ 2)\ (4\ 6\ 5) = f_3^2$$

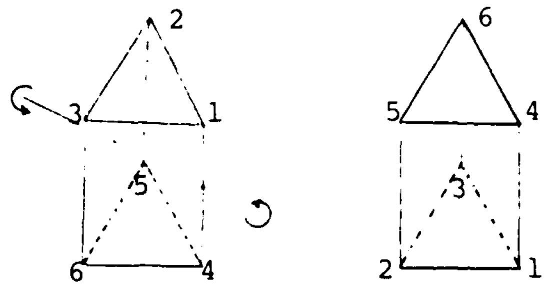
Rotating 240° around a vertical axe that goes through the center of symmetry of both triangular faces,



$$\begin{pmatrix}1&2&3&4&5&6\\2&3&1&5&6&4\end{pmatrix} = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}\begin{pmatrix}4&5&6\\5&6&4\end{pmatrix} = (1\ 2\ 3)\ (4\ 5\ 6) = f_3^2$$

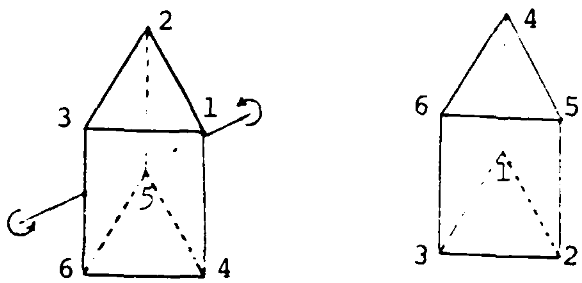
The prism also has three horizontal axes of symmetry going through the middle point of each rectangular face to the middle point of the respective opposite edge.

Rotating 180° around the horizontal axe that goes through the middle point of the face, 2365 to the middle point of edge 14,



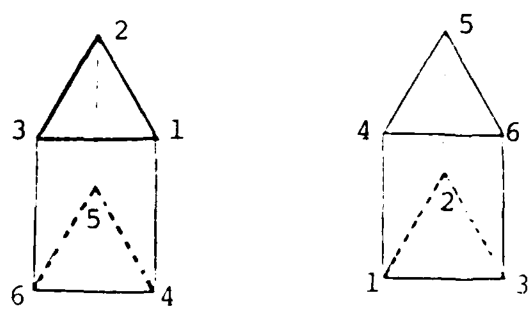
$$\begin{pmatrix}1&2&3&4&5&6\\4&6&5&1&3&2\end{pmatrix} = \begin{pmatrix}1&4\\4&1\end{pmatrix}\begin{pmatrix}2&6\\6&2\end{pmatrix}\begin{pmatrix}3&5\\5&3\end{pmatrix} = (1\ 4)\ (2\ 6)\ (3\ 5) = f_2^3$$

Rotating 180° around the horizontal axe that goes from the middle point of the face 1254 to the middle point of edge 36,



$$\begin{pmatrix}1&2&3&4&5&6\\5&4&6&2&1&3\end{pmatrix} = \begin{pmatrix}1&5\\5&1\end{pmatrix}\begin{pmatrix}2&4\\4&2\end{pmatrix}\begin{pmatrix}3&6\\6&3\end{pmatrix} = (1\ 5)\ (2\ 4)\ (3\ 6) = f_2^3$$

Rotating 180° around the horizontal axe that goes from the middle point of the face 1364 to the middle point of edge 25,



$$\binom{1\ 2\ 3\ 4\ 5\ 6}{6\ 5\ 4\ 3\ 2\ 1} = \binom{1\ 6}{6\ 1} \binom{2\ 5}{5\ 2} \binom{3\ 4}{4\ 3} = (1\ 6)\ (2\ 5)\ (3\ 4) = f_2^3$$

There are a total of 6 rotations performed, which produce the following structures:

$$f_1^6 + f_3^2 + f_3^2 + f_2^3 + f_2^3 + f_2^3 = f + 2f_3^2 + 3f_2^3$$

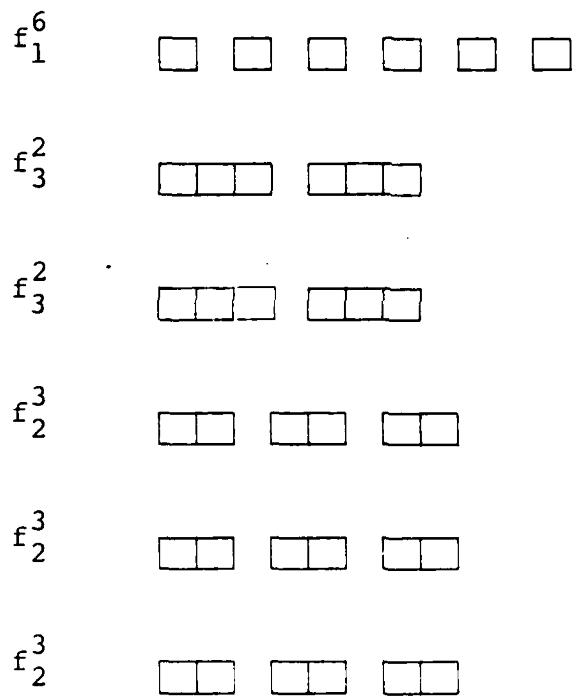
So, the cycle index for the regular triangle prism

$$= \frac{f_1^6 + 2f_3^2 + 3f_2^3}{6}$$

Recall here again that the cycle index is really the unit because $f_1^6 + 2f_3^2 + 3f_2^3$ has only a symbolic value for indicating the interrelation between the original and final situations of the six vertices. Thus, the cycle index condenses the following information.

$$\frac{1}{6} \cdot \begin{array}{l} \quad\ 1 \text{ time } (1)\ (2)\ (3)\ (4)\ (5)\ (6) \\ + \ 1 \text{ time } (132)\ (465) \\ + \ 1 \text{ time } (123)\ (456) \\ + \ 1 \text{ time } (14)\ (26)\ (35) \\ + \ 1 \text{ time } (15)\ (24)\ (36) \\ + \ 1 \text{ time } (16)\ (25)\ (34) \end{array} \qquad [1]$$

Later on, when the cycle index is expanded according to the specific number of variables playing a role in the given problem, one has to compute the different arrangements of those variables into all the cycles mentioned in expression [1]. Graphically, those arrangements of variables represent a problem equivalent to the one consisting in filling, with repeated selection, the following boxes:



The constraints in solving this problem are:

1. Repeated selection of the two colored beads is <u>not</u> allowed.

2. *The joined boxes represent cycles, and therefore the same type of colored beads have to fill them.*

2. Pattern inventory. There are three variables in this problem:

a. color a

b. color b

c. no color c

Thus, the pattern inventory function will consist of the replacement of the following convenient expression into the cycle index formula,

$$f_q^p \longrightarrow (a^q + b^q + c^q)^p .$$

$$\text{Cycle index for the regular triangular prism} = \frac{f_1^6 + 2f_3^2 + 3f_2^3}{6}$$

$$\text{Expanded cycle index for the regular triangular prism} = \frac{(a+b+c)^6 + 2(a^3+b^3+c^3)^2 + 3(a^2+b^2+c^2)^3}{6}$$

The development of the whole expanded cycle index produces a result valid for attaching colored beads to the vertices with repeated selection. But in this problem there is no chance for repeated selection because there are only two beads available. Thus, the only valid structure here is $a\,b\,c^4$ (1 bead of color a, 1 bead of color b, and three vertices with no beads).

The term $f_1^6 = (a+b+c)^6$ produces the structure $a\,b\,c^4$, which coefficient can be attained applying the Leibniz Formula,

$$(a+b+c)^6 = \Sigma \frac{6!}{1!1!4!} a\,b\,c^4 = 30\ a\,b\,c^4$$

$$1 + 1 + 4 = 6$$

The rest of the terms within the cycle index,

$2f_3^2 = 2(a^3 + b^3 + c^3)^2$ and $3f_2^3 = 3(a^2 + b^2 + c^2)^3$

do not produce the structure abc^4 obviously. The value of the expanded cycle index will be something like:

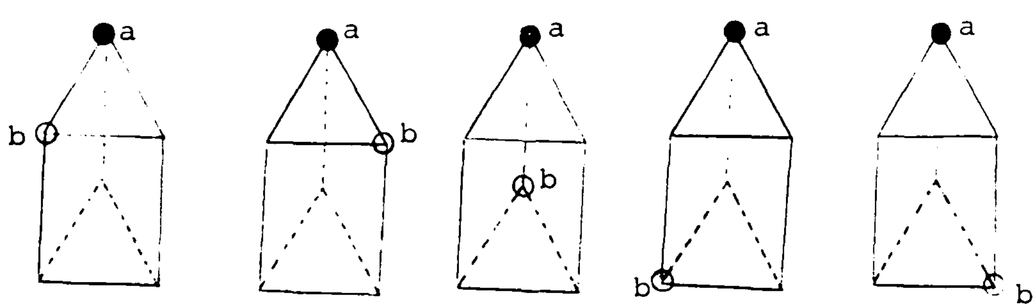
$$\frac{1}{6}\ [a^6 + b^6 + c^6 + 30\ a\,b\,c^4 + \ldots + 2(a^3 + b^3 + c^3)^2 + 3(a^2 + b^2\ c^2)],$$

from where the only valid solution, that accomplishes the constraints, is taken,

$$\frac{1}{6} \cdot 30\,a\,b\,c^4 = 5\,a\,b\,c^4$$

Thus, the solution is 5 different ways.

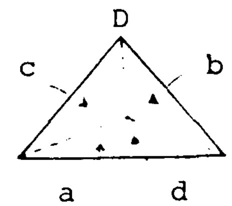
The same solution can be graphically computed:



Solution: 5 different ways

Problem No. 4. "Find the number of ways of painting the four faces a, b, c, and d of a isosceles tetrahedron with two colors of paints, x and y" (Liu, 1968, 150).

Solution.



Explanation about the initial presentation of the tetrahedron according to the spectator point of view:

Face a is located just in front of the reader.

Face b is located at the back right hand side of the pyramid.

Face c is located at the back left hand side of the pyramid.

Face d is the base of the pyramid.

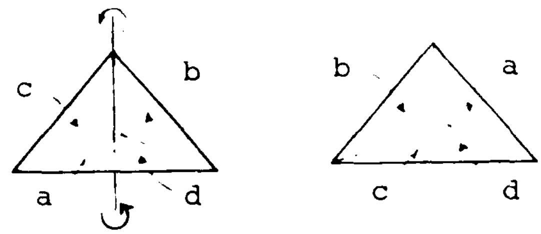
Faces a, b, and c are equal; face d is not.

Notice, on the other hand, that the isosceles tetrahedron only has one axis of symmetry that vertically goes through vertex D on the top and the center of gravity (baricenter) of the opposite face d at the base. That axis of symmetry is superimposed on to the pyramid height. During rotations face d will remain in its initial place.

1. Cycle index. No rotation. Initial position. Identity substitution.

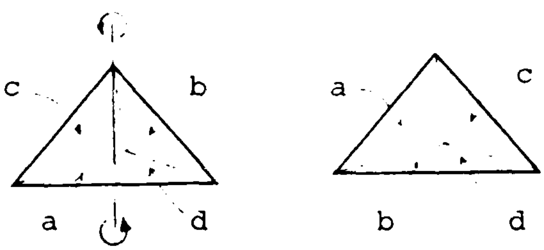
$$\begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix}\begin{pmatrix} b \\ b \end{pmatrix}\begin{pmatrix} c \\ c \end{pmatrix}\begin{pmatrix} d \\ d \end{pmatrix} = (a)(b)(c)(d) = f_1^4$$

Rotating $120° = \frac{360°}{3}$ around the tetrahedron height,



$$\begin{pmatrix} a & b & c & d \\ c & a & b & d \end{pmatrix} = \begin{pmatrix} a & c & b \\ c & b & a \end{pmatrix} \begin{pmatrix} d \\ d \end{pmatrix} = (acb)(d) = f_3^1 \cdot f_1^1$$

Rotating 240° around the tetrahedron height,



$$\begin{pmatrix} a & b & c & d \\ b & c & a & d \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}\begin{pmatrix} d \\ d \end{pmatrix} = (abc)\ (d) = f_3^1 \cdot f_1^1$$

Total of rotations performed: 3 (cycle index denumerator). Types of circular substitutions or cycles made with those 3 rotations:

$$f_1^4 + f_3^1 f_1^1 + f_3^1 f_1^1 = f_1^4 + 2f_1^1 f_3^1 \ (\text{cycle index numerator})$$

Thus, the cycle index is

Cycle index for the isosceles tetrahedron $= \dfrac{f_1^4 + 2f_1^1 f_3^1}{3}$

2. Pattern inventory. There are two variables in this problem:

a. color x

b. color y

and both colors can be chosen with repeated selection. Notice the difference with the regular triangular prism problem, where there was not any chance for repeated selection, and the number of variables was three (color a, color b, no color).

The pattern inventory function will consist in the replacement of the following convenient expression into the cycle index formula,

$$f_q^p \longrightarrow (x^q + y^q)^p$$

Cycle index for the isosceles tetrahedron $= \dfrac{f_1^4 + 2f_1^1 f_3^1}{3}$

$$= \frac{1}{3} \cdot [(x+y)^4 + 2(x+y)(x^3+y^3)]$$

$$= \frac{1}{3}\ [(x+y)^2(x+y)^2+2x^4+2xy^3+2x^3y+2y^4]$$

$$= \frac{1}{3}[(x^2+y^2+2xy)^2+2x^4+2x^3y+2xy^3+2y^4]$$

$$= \frac{1}{3}\cdot(x^4+y^4+2x^2y^2+4x^2y^2+4x^3y+4xy^3+2x^4+2x^3y+2xy^3+2y^4)$$

$$= \frac{1}{3}\ (x^4+2x^4+4x^3y+2x^3y+2x^2y^2+4x^2y^2+4xy^3+2xy^3+y^4+2y^4)$$

$$= \frac{1}{3}\ (3x^4+6x^3y+6x^2y^2+6xy^3+3y^4)$$

$$= x^4 + 2x^3y + 2x^2y^2 + 2xy^3 + y^4 \qquad [1]$$

There are here 8 possibilities of painting the four faces of the pyramid with repeated selection:

	Face a	Face b	Face c	Face d
1st	x	x	x	x
2nd	x	x	x	y
3rd	x	x	y	x
4th	x	x	y	y
5th	y	y	x	x
6th	x	y	y	y
7th	y	y	y	x
8th	y	y	y	y

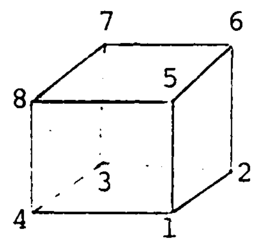
The same result is obtained adding the coefficient of [1]:

$$1 + 2 + 2 + 2 + 1 = 8.$$

<u>Problem No. 5</u>. "Find the distinct ways of painting the eight vertices of a cube with two colors x and y" (Liu, 1968:151).

<u>Solution</u>.

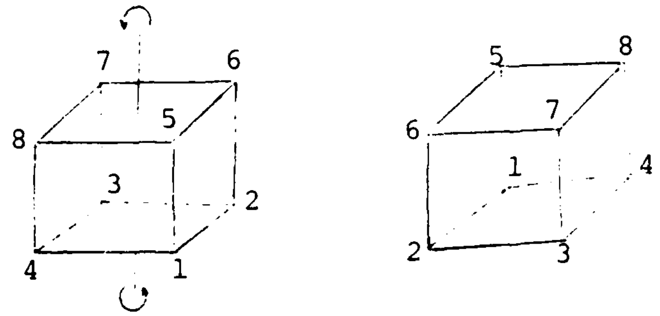
1. Cycle index. No rotation. Initial position. Identity substitution.



$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\end{pmatrix} = \binom{1}{1}\binom{2}{2}\binom{3}{3}\binom{4}{4}\binom{5}{5}\binom{6}{6}\binom{7}{7}\binom{8}{8}$$

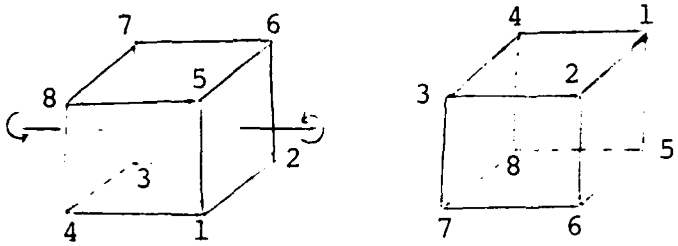
$$= (1)\ (2)\ (3)\ (4)\ (5)\ (6)\ (7)\ (8) = f_1^8$$

Three 180° rotations around axes connecting the centers of opposite faces,



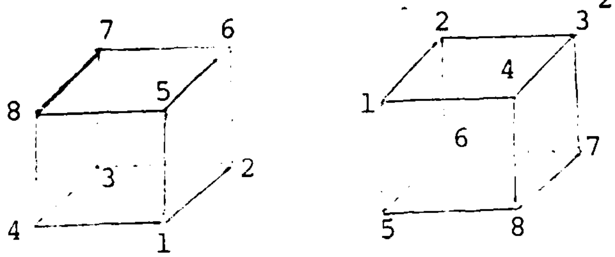
$$\begin{pmatrix}1&2&3&4&5&6&7&8\\3&4&1&2&7&8&5&6\end{pmatrix} = \begin{pmatrix}1&3\\3&1\end{pmatrix}\begin{pmatrix}2&4\\4&2\end{pmatrix}\begin{pmatrix}5&7\\7&5\end{pmatrix}\begin{pmatrix}6&8\\8&6\end{pmatrix}$$

$$= (13)\ (24)\ (57)\ (68) = f_2^4$$



$$\begin{pmatrix}1&2&3&4&5&6&7&8\\6&5&8&7&2&1&4&3\end{pmatrix} = \begin{pmatrix}1&6\\6&1\end{pmatrix}\begin{pmatrix}2&5\\5&2\end{pmatrix}\begin{pmatrix}3&8\\8&3\end{pmatrix}\begin{pmatrix}4&7\\7&4\end{pmatrix}$$

$$= (16)\ (25)\ 38)\ (47) = f_2^4$$

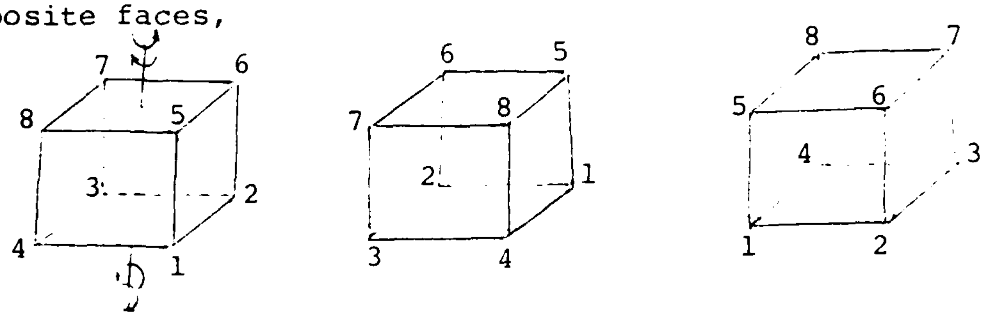


$$\begin{pmatrix}1&2&3&4&5&6&7&8\\8&7&6&5&4&3&2&1\end{pmatrix} = \begin{pmatrix}1&8\\8&1\end{pmatrix}\begin{pmatrix}2&7\\7&2\end{pmatrix}\begin{pmatrix}3&6\\6&3\end{pmatrix}\begin{pmatrix}4&5\\5&4\end{pmatrix}$$

$$= (18)\ (27)\ (36)\ (45) = f_2^4$$

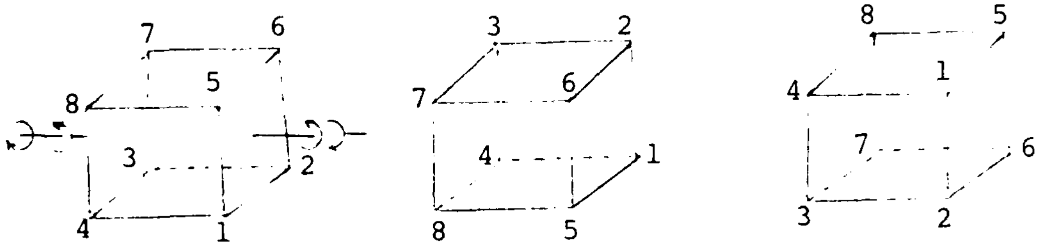
There are $3f_2^4$ cycles.

Six ± 90° rotations around lines connecting the center of opposite faces,



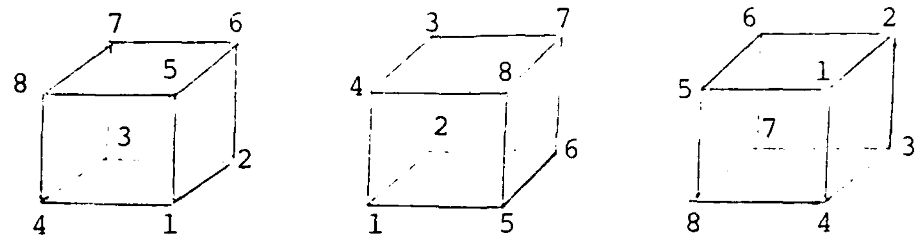
$$\begin{pmatrix}1&2&3&4&5&6&7&8\\4&1&2&3&8&5&6&7\end{pmatrix} = \begin{pmatrix}1&4&3&2\\4&3&2&1\end{pmatrix}\begin{pmatrix}5&8&7&6\\8&7&6&5\end{pmatrix} = (1\ 4\ 3\ 2)(5\ 8\ 7\ 6) = f_4^2$$

$$\begin{pmatrix}1&2&3&4&5&6&7&8\\2&3&4&1&6&7&8&5\end{pmatrix} = \begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}5&6&7&8\\6&7&8&5\end{pmatrix} = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8) = f_4^2$$



$$\begin{pmatrix}1&2&3&4&5&6&7&8\\5&1&4&8&6&2&3&7\end{pmatrix} = \begin{pmatrix}1&5&6&2\\5&6&2&1\end{pmatrix}\begin{pmatrix}3&4&8&7\\4&8&7&3\end{pmatrix} = (1\ 5\ 6\ 2)(3\ 4\ 8\ 7) = f_4^2$$

$$\begin{pmatrix}1&2&3&4&5&6&7&8\\2&6&7&3&1&5&8&4\end{pmatrix} = \begin{pmatrix}1&2&6&5\\2&6&5&1\end{pmatrix}\begin{pmatrix}3&7&8&4\\7&8&4&3\end{pmatrix} = (1\ 2\ 6\ 5)(3\ 7\ 8\ 4) = f_4^2$$

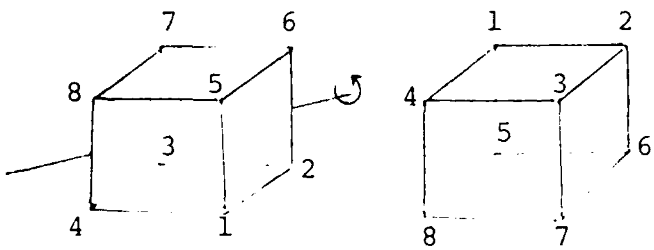


$$\begin{pmatrix}1&2&3&4&5&6&7&8\\5&6&2&1&8&7&3&4\end{pmatrix} = \begin{pmatrix}1&5&8&4\\5&8&4&1\end{pmatrix}\begin{pmatrix}2&6&7&3\\6&7&3&2\end{pmatrix} = (1\ 5\ 8\ 4)(2\ 6\ 7\ 3) = f_4^2$$

$$\begin{pmatrix}1&2&3&4&5&6&7&8\\4&3&7&8&1&2&6&5\end{pmatrix} = \begin{pmatrix}1&4&8&5\\4&8&5&1\end{pmatrix}\begin{pmatrix}2&3&7&6\\3&7&6&2\end{pmatrix} = (1\ 4\ 8\ 5)(2\ 3\ 7\ 6) = f_4^2$$

There are $6f_4^2$ cycles.

Six 180° rotations around lines connecting the midpoint of opposite edges,



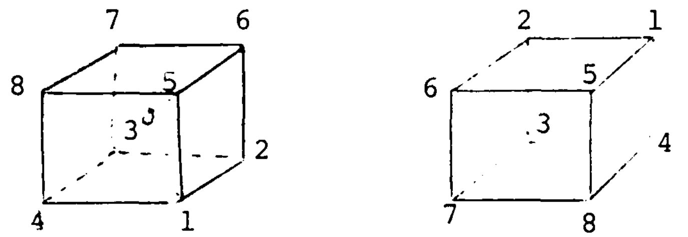
$$\begin{pmatrix}1&2&3&4&5&6&7&8\\7&6&5&8&3&2&1&4\end{pmatrix} = \begin{pmatrix}1&7\\7&1\end{pmatrix}\begin{pmatrix}2&6\\6&2\end{pmatrix}\begin{pmatrix}3&5\\5&3\end{pmatrix}\begin{pmatrix}4&8\\8&4\end{pmatrix}$$

$$= (17)\ (26)\ (35)(48)$$

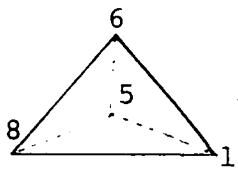
$$= f_2^4$$

There are $6f_2^4$ cycles.

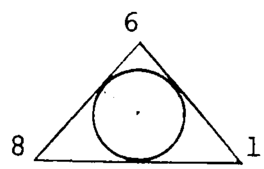
Eight ±120° rotations around lines connecting opposite vertices (the diagonals of the cube)



Notice the equilateral triangle  . The pyramid



rotates around point 5 and its base 1 6 8.



$$\frac{360}{3} = 120°$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 4 & 3 & 7 & 5 & 1 & 2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 8 & 6 \\ 8 & 6 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 7 \\ 4 & 7 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

$$= (3)(5)(186)(247)$$

$$= f_1^2 \cdot f_3^2$$

There are $8f_1^2f_3^2$ cycles.

Types of circular substitutions or cycles made with these rotations:

$$f_1^8 + 3f_2^4 + 6f_4^2 + 6f_2^4 + 8f_1^2f_3^2$$

Total number of rotations performed:

$$1 + 3 + 6 + 8 = 24 \text{ rotations.}$$

Without numerical calculations one can see that there are 24 different positions (or rotations to be performed) for the cube, by observing that there are six faces any of which can be positioned at the top; and that for each choice there are then four faces any of which can be positioned at the front.

$$\text{Cycle index for the cube} = \frac{f_1^8 + 9f_2^4 + 6f_4^2 + 8f_1^2f_3^2}{24}$$

2. Pattern inventory. There are two variables in this problem, where repeated selection of colors is allowed:

a. color x

b. color y

Thus, the pattern inventory is $f_q^p \longrightarrow (x^q+y^q)^p$

Expanded cycle index for the cube $= \dfrac{(x+y)^8+9(x^2+y^2)+6(x^4+y^4)^2+8(x+y)^2(x^3+y^3)^2}{24}$

All possible combinations of colors represented in the expanded cycle index are valid. As the problem only requests the number of distinct ways of painting, thus, colors x and y can be pondered equal 1 (both are equally distributed in the whole tableau of different arrangements).

For $\quad x = y = 1,$

$$\frac{2^8 + 9\cdot 2^4 + 6\cdot 2^2 + 8\cdot 2^2\cdot 2^2}{24} = 23 \text{ ways}$$

After this problem has been already solved, the reader should have noticed the practical use of the notation f_q^p just in order to represent and collect homogeneous cycles after the rotations have been performed. By applying this notation, a lot of repetitive and tedious calculations can be saved, because the symbolism f_q^p allows one to group together all the homogeneous cycles that follow the same structure, even though they have different contents. For instance,

$$mf_q^p + nf_q^p = (m+n)f_q^p = Mf_q^p$$

Those M cycles, grouped together because of their common structure, have different contents, which are ignored at a first sight. But the application of the pattern inventory to every term f_q^p in the cycle index

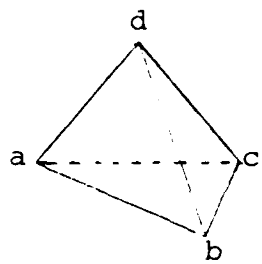
returns its full sense to every singular cycle and to every item, vertex, face, side, edge, variable, etc.

<u>Problem No. 6</u>. Given a regular tetrahedron and four different colors <u>a</u>, <u>b</u>, <u>c</u>, and <u>d</u> calculate:

a. The number of different configurations that can be obtained by painting the four vertices with the four colors, with repeated selection.

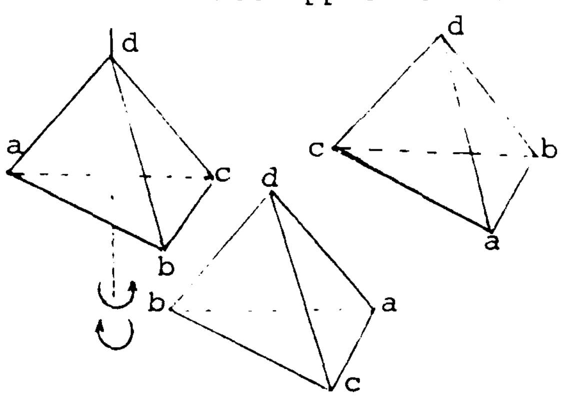
b. The distribution of color <u>a</u> in the different configurations obtained in 1.

1. Cycle index. No rotation. Initial arrangement. Identity substitution.



$$\begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix} = (a\ b\ c\ d) = (a)(b)(c)(d) = f_1^4$$

±120° rotations around lines connecting a vertex and the center of its opposite face.



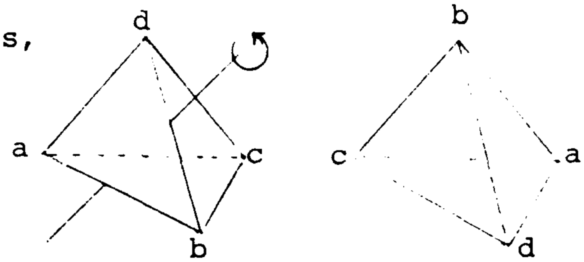
$$\begin{pmatrix} a & b & c & d \\ c & a & b & d \end{pmatrix} = (a\ c\ b)(d) = f_3^1 f_1^1$$

$$2f_3^1 f_1^1$$

$$\begin{pmatrix} a & b & c & d \\ b & c & a & d \end{pmatrix} = (a\ b\ c)(d) = f_3^1 f_1^1$$

As the same structure happens four times, considering the four vertices on the top, $4 \times 2f_3^1 f_1^1 = 8\ f_1^1 f_3^1$

180° rotations around lines connecting the midpoints of opposite edges,



$$\begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix} = (a\ c)\ (b\ d) = f_2^2$$

As the same structure happens three times, $3 x f_2^2 = 3f_2^2$,

thus,

$$\text{Cycle index for regular tetrahedron} = \frac{f_1^4 + 8f_1^1f_3^1 + 3f_2^2}{1 + 8 + 3} = \frac{f_1^4 + 8f_1^1f_3^1 + 3f_2^2}{12}$$

The same result can be obtained observing that the four faces can serve as the tetrahedron base, and that for each base there are three possible edges at the front.

So, 4 x 3 = 12 different configurations.

There are four variables: a, b, c, d.

Pattern inventory $f_q^p = (a^q + b^q + c^q + d^q)^p$

$$\text{Expanded cycle index for regular tetrahedron} = \frac{(a+b+c+d)^4+8(a+b+c+d)(a^3+b^3+c^3+d^3)+3(a^2+b^2+c^2+d^2)^2}{12}$$

Weighting a = b = c = d = 1 ,

number of different ways for painting the vertices of the regular tetrahedron, with repeated selection =

$$= \frac{1}{12} \cdot (4^4+8\cdot 4\cdot 4+3\cdot 4^2) = \frac{1}{12}(256+128+48) = \frac{432}{12} = 36$$

2. Weighting a = a; b = c = d = 1, the pattern inventory is $f_q^p = (a^q + 1 + 1 + 1)^p$, and the

$$\text{Expanded Cycle Index} = \frac{(a+3)^4+8(a+3)(a^3+3)+3(a^2+3)^2}{12}$$

$$= \frac{(a^2+6a+9)^2+8(a^4+3a^3+3a+9)+3(a^4+6a^2+9)}{12}$$

$$= \frac{1}{12}(a^4+12a^3+36a^2+18a^2+108a+81+8a^4+24a^3+24a$$

$$+72+3a^4+18a^2+27)$$

$$= \frac{1}{12}(12a^4+36a^3+72a^2+132a+180)$$

$$= a^4 + 3a^3 + 6a^2 + 11a + 15.$$

Distribution:

1 arrangement containing four vertices painted with a

3 arrangements containing three vertices painted with a

6 arrangements containing two vertices painted with a

11 arrangements containing one vertex painted with a

15 arrangements containing zero vertices painted with a

---

36 total arrangements, 21 of which (1+3+6+11) contain color a.

V. Conclusions and Recommendations for Further Research

Conclusions

The capital characteristics of the number applied to a set of elements are two:

1. That number is independent of the order in which the elements are taken into account;

2. That number is independent of the nature of the elements it compiles.

But when combinatorics concepts are employed for computing what number has to be assigned to a set of elements, the first of these two characteristics could be modified in the sense that the number to be given can be dependent or independent of the elements' order. Such distinction makes the difference between variations, where order of the elements is relevant, and combinations, where order is irrelevant.

Besides the sense of order as a variable for computing the elements of a set, combinatorics can deal with another variable: the possibility of selecting all the elements in a repetitive manner. Such distinction makes the difference between ordinary variations and combinations versus variations and combinations with repeated selection.

Recall at this point that ordinary permutations are a particular case of ordinary variations, and recall also that permutations with repetition play a key role for rejecting duplicate arrangements of elements.

Moreover, most of those calculations have only been researched in one dimensional space, leaving an open field for future studies. A modest attempt was made in two dimensions (circular permutations) and three dimensions (substitutions).

It is important to remark here that equivalent problems arrive to distinct solutions when they are solved under different dimensional conditions. To greater degrees of freedom, corresponds a more simple answer, and vice versa. Thus, when the number of dimensions increases, the number of solutions for any particular problem decreases. For instance, compare the three distinct results obtained when _n_ different colored beads are arranged in one, two or three dimensions:

1. Arrangements along a row, $P_n=n!$ different arrangements;

2. Arrangements around a circular table, $PC_n = P_{n-1} = (n-1)!$ different arrangements.

3. Arrangements made by stringing the beads into a necklace, $\frac{PC_n}{2} = \frac{(n-1)!}{2}$ different arrangements.

The conclusion goes against normal human intuition:

$n!$	$>$	$(n-1)!$	$>$	$\frac{(n-1)!}{2}$
One dimension		Two dimensions		Three dimensions

On the other hand, consider the basic alternatives that could be present at solving any specific problem:

1. Ordinary selection (variations, permutations, combinations)

2. Repeated selection (variations, combinations)

3. Repetition (permutations)

No other research effort (as far as one knows) has been conducted in order to solve problems where the three alternatives are simultaneously presented. For example, given an original set of m beads, in which some of those beads are unique and, therefore, they only can be used one time; some other beads are repeated a finite number of times (e.g., α times red, β times green, ..., etc.) and they can be repeated that finite number as a maximum; and the rest of the beads are repeated indefinitely and can be used with no restriction. A question that could well follow the former data could be: in how many different ways could those beads be attached to the n vertices of a regular polyhedron? Imagine the complexity of problems of this nature, moreover in three dimensions, and even

more if m > n, because the pattern inventory function cannot be discerned when the variables do not obey the same law.

Nevertheless, a new approach for teaching combinatorics has been developed by the present thesis effort as a remedy to the fact that previous presentations leave a great deal to be desired pedagogically, not only because most textbooks use very difficult languages and notations for beginners, but also because they do not place enough emphasis on visualization and fail to show the relationships between general and subordinate concepts. Thus, a partial solution has been found for promoting meaningful learning of combinatorial concepts, giving to them a touch of freshness and relevance. It is true that one has not been able to solve every facet of the problem; that would require at least a lifetime of study and research, but it has been demonstrated that it is solvable.

Recommendations for Further Research

One regrets the lack of time for continuing deeper theoretical research through the fascinating world of combinatorics. Far from being discouraged by this fact, this research effort has opened the door to at least two new research efforts:

1. Explore the characteristics and structure of those problems already presented in the first part of this

chapter, dealing with situations in two and three dimensions where the possibility for the simultaneous occurrence of ordinary selection, repeated selection and repetition exists.

2. Investigate ways that computer graphics and expert systems can be used to facilitate the accession and employment of the conceptual map presented in Chapter IV in order to enhance the visualization and computational capabilities of the combinatorial analyst.

Finally, one hopes that this modest attempt to advance the theory and pedagogy of combinatorics has at least made some contribution to the science.

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Vita

Major Pedro Rodriguez-Pascual was born on 28 September 1943 in Granada, Spain. He graduated from high school in his home town in 1960, and attended a pre-military college from 1960 to 1963. He finally entered the Air Force Academy, from which he received a Masters degree in Military Sciences in 1967. Upon graduation, he served in Sevilla (Tablada AFB), Logrono (Agoncillo AFB), Las Palmas (Gando AFB) and Madrid (AF Headquarters). From 1973 to 1977 he attended the University of Madrid, from which he received a Masters degree in Law and, subsequently, in 1983 he received a Ph.D. in Law. From 1977 to 1986 he served in Sacramento (McClellan AFB) and Madrid (AF Senior Staff) until entering the School of Systems and Logistics, Air Force Institute of Technology, in May 1986.

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Block 19--Abstract

The purpose of this study was to construct a new methodology for teaching combinatorics based on Doctor Ausubel's theory about meaningful learning. The key idea in Ausubel's theory is that if learning has to be meaningful, then the learner has to have subsuming or anchoring concepts in his cognitive structure.

Combinatorics has typically been one of those subjects the students have more difficulty in understanding. This phenomenon happens because previous presentations of combinatorics leave a great deal to be desired pedagogically, and do not place enough emphasis on visualization. As a result, students use to learn course materials in a rote manner, and find little motivation for such learning activities.

A prescription has been found to remedy such pathology. A conceptual map, rather than a typically organized hierarchy of concepts, has been developed. The conceptual map interrelates the main and subordinate concepts in a cyclical manner, in a repetitive way, in a gradual and smooth progress, to enable the reader to assimilate ideas meaningfully.

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